NONLINEAR FRACTIONAL SCHRÖDINGER EQUATIONS COUPLED BY POWER–TYPE NONLINEARITIES

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Abstract. In this work we study the following class of systems of coupled nonlinear fractional Schrödinger equations,

$$\left\{\begin{array}{ll}(-\Delta)^s u_1+\lambda_1 u_1=\mu_1|u_1|^{2p-2}u_1+\beta|u_2|^p|u_1|^{p-2}u_1 & \text{in } \mathbb{R}^N,\\ (-\Delta)^s u_2+\lambda_2 u_2=\mu_2|u_2|^{2p-2}u_2+\beta|u_1|^p|u_2|^{p-2}u_2 & \text{in } \mathbb{R}^N,\end{array}\right.$$

where $u_1,\, u_2\in W^{s,2}(\mathbb{R}^N)$, with $N=1,\,2,\,3$; $\lambda_j,\,\mu_j>0$, $j=1,2$, $\beta\in\mathbb{R}$, $p\geq 2$ and $\frac{p-1}{2p}N<s<1$. We prove the existence of positive radial bound and ground state solutions provided the parameters β,p,λ_j,μ_j, $(j=1,\,2)$ satisfy appropriate conditions. We also study the previous system with m-equations,

$$(-\Delta)^s u_j+\lambda_j u_j=\mu_j|u_j|^{2p-2}u_j+\sum_{\substack{k=1\\k\neq j}}^{m}\beta_{jk}|u_k|^p|u_j|^{p-2}u_j,\quad u_j\in W^{s,2}(\mathbb{R}^N);\ j=1,\ldots,m$$

where $\lambda_j,\,\mu_j>0$ for $j=1,\ldots,m\geq 3$, the coupling parameters $\beta_{jk}=\beta_{kj}\in\mathbb{R}$ for $j,k=1,\ldots,m$, $j\neq k$. We prove similar results as for $m=2$, depending on the values of the parameters $p,\beta_{jk},\lambda_j,\mu_j$.

Introduction

In this work we study the existence of positive solutions to the following system of coupled nonlinear fractional Schrödinger (NLFS) equations,

$$\left\{\begin{array}{ll}(-\Delta)^s u_1+\lambda_1 u_1=\mu_1|u_1|^{2p-2}u_1+\beta|u_2|^p|u_1|^{p-2}u_1 & \text{in } \mathbb{R}^N,\\ (-\Delta)^s u_2+\lambda_2 u_2=\mu_2|u_2|^{2p-2}u_2+\beta|u_1|^p|u_2|^{p-2}u_2 & \text{in } \mathbb{R}^N,\end{array}\right. \tag{1}$$

where $u_j\in W^{s,2}(\mathbb{R}^N)$ with $N=1,\,2,\,3$; $\lambda_j,\,\mu_j>0$ for $j=1,2$, the coupling factor $\beta\in\mathbb{R}$, $p\geq 2$ and $\frac{p-1}{2p}N<s<1$. We also study the previous system with m-equations,

$$(-\Delta)^s u_j+\lambda_j u_j=\mu_j|u_j|^{2p-2}u_j+\sum_{\substack{k=1\\k\neq j}}^{m}\beta_{jk}|u_k|^p|u_j|^{p-2}u_j, \tag{2}$$

The authors are partially supported by MCIN/AEI/10.13039/501100011033, research project PID2019-106122GB-I00. They are also partially supported by the Madrid Government (Comunidad de Madrid) under the Multiannual Agreement with UC3M in the line of Excellence of University Professors (EPUC3M23), and in the context of the V PRICIT (Regional Programme of Research and Technological Innovation).

The talk at the 8IMM 2022 has been given by Alejandro Ortega.

with $u_j \in W^{s,2}(\mathbb{R}^N)$, λ_j, $\mu_j > 0$, $j = 1, \dots, m \geq 3$, the coupling parameters $\beta_{jk} = \beta_{kj} \in \mathbb{R}$ for $j, k = 1, \dots, m$, $j \neq k$. We prove similar results as for (1), depending on the values of the parameters $p, \beta_{jk}, \lambda_j, \mu_j$.

Problems like (1) have been widely investigated with the classical Laplacian ($s = 1$) in the last years so it is complicated to give a complete list of references. We refer, among others, to [2, 3, 5, 6, 11, 13, 15, 16, 18, 24, 25, 26, 29, 30, 32, 34, 36, 37] and references therein. It is well known that solutions of (1), at least for the classical case $s = 1$, are related to the solitary waves of the Gross-Pitaevskii equations, which have applications in many physical models, such as in nonlinear optics (cf. [1, 27, 28]) and in multi-species Bose-Einstein condensates (cf. [10, 31]). Actually, a planar light beam propagating in the z direction in a non-linear medium, can be described by a vector NLS equation like

$$i\,\mathbf{E}_z + \mathbf{E}_{xx} + \kappa|\mathbf{E}|^2\mathbf{E} = 0,$$

where i, $\mathbf{E}(x,z)$ denote the imaginary unit and the complex envelope of an Electric field, respectively. If $\mathbf{E}$ is the sum of two right- and left-hand polarized waves a_1E_1 and a_2E_2, $a_j \in \mathbb{R}$, then, assuming $\kappa = 1$, solitary wave solutions $E_j(z,x) = e^{\mathrm{i}\lambda_j z}u_j(x)$, where $\lambda_j > 0$ and $u_j(x)$ are real valued functions, provide us with the system

$$\left\{ \begin{array}{l} -u_1'' + \lambda_1 u_1 = (a_1^2u_1^2 + a_2^2u_2^2)u_1, \\ -u_2'' + \lambda_2 u_2 = (a_1^2u_1^2 + a_2^2u_2^2)u_2. \end{array} \right. \tag{3}$$

If we take the coupling factor $a_1^2 = a_2^2 := \beta$ as a parameter and let the coefficients of u_j^3, namely $\mu_j > 0$, to be different, then (3) corresponds to (1) with $N = 1$, $s = 1$ and $p = 2$. Similarly, looking for solitary wave solutions for the NLFS equation in $\mathbb{R}^N$,

$$i\,\mathbf{E}_z - (-\Delta)^s\mathbf{E} + \kappa|\mathbf{E}|^2\mathbf{E} = 0,$$

one arrives to system (1) with $p = 2$. We point out that this type of nonlocal diffusion involving the fractional Laplacian $(-\Delta)^s$ arises in several physical phenomena like flames propagation and chemical reactions, population dynamics, geophysical fluid dynamics, as well as in probability, American option in finance or in α-stable Lévy processes (with $\alpha = 2s$) (cf. [4, 7, 14, 35]). Here we are interested in systems of coupled NLFS equations involving the so called fractional Schrödinger operator, $(-\Delta)^s + \lambda\,\mathrm{Id}$, (cf. [17, 22, 23]).

Our main aim is then to give a classification of positive solutions of (1) and also for the system with m-equations (2). Precisely, we will prove the following.

-Existence of positive radial ground states under the following hypotheses:

- $p = 2$ and the coupling coefficient $\beta > \Lambda'$; see Theorem 3.2,
- $p \geq 2$ and the coupling coefficient β satisfying hypothesis $(\mathcal{H})$; see Theorem 4.3.

-Existence of radial bound states when:

- $p = 2$ and $\beta < \Lambda$; see Theorem 3.3-(i) which are positive provided $\beta > 0$,
- $p > 2$ and $\beta \in \mathbb{R}$; see Theorem 3.3-(ii), which are positive when $\beta > 0$,
- $p \geq 2$ and $\beta \sim 0$; see Theorem 3.3-(iii) and Theorem 4.4. The radial bound states are positive for $\beta > 0$. We also prove a bifurcation result.

1. Preliminaries and Notation

Given $0 < s < 1$, the nonlocal operator $(-\Delta)^s$ in $\mathbb{R}^N$ is defined on the Schwartz class of functions $g \in \mathcal{S}$ through the Fourier transform,

$$[(-\Delta)^s g]^\wedge(\xi) = (2\pi|\xi|)^{2s}\,\widehat{g}(\xi),$$

or via the Riesz potential, (cf. [21, 33]). Observe that $s = 1$ corresponds to the classical Laplacian. There is another way of defining this operator. In fact, for $s = \frac{1}{2}$, the square root of the Laplacian acting on a function u in the whole space $\mathbb{R}^N$, can be calculated as the normal derivative of its harmonic extension to the upper half-space $\mathbb{R}^{N+1}_+$, this is so-called Dirichlet to Neumann operator. Based on this idea, Caffarelli and Silvestre (cf. [9]) proved that $(-\Delta)^s$ can be realized in a local way by using the s-harmonic extension.

More precisely, given u a regular function in $\mathbb{R}^N$, we define its s-harmonic extension to the upper half-space $\mathbb{R}^{N+1}_+$, denoted by $w = E_s[u]$, as the solution to the problem

$$\begin{cases} -\mathrm{div}(y^{1-2s}\nabla w) = 0 & \text{in } \mathbb{R}^{N+1}_+, \\ w = u & \text{on } \mathbb{R}^N \times \{y = 0\}. \end{cases}$$

The key point of the s-harmonic extension comes from the following identity (cf. [9]),

$$-\kappa_s \lim_{y\to 0^+} y^{1-2s}\frac{\partial w}{\partial y}(x,y) = (-\Delta)^s u(x),$$

with $\kappa_s = 2^{2s-1}\frac{\Gamma(s)}{\Gamma(1-s)}$. The above Dirichlet-Neumann procedure provides a formula for the fractional Laplacian in $\mathbb{R}^N$, equivalent to that obtained using the Fourier transform. In this case, the s-harmonic extension and the fractional Laplacian have explicit expressions in terms of the Poisson and the Riesz kernels respectively (cf. [8]),

$$w(x,y) = c_{N,s}\, y^{2s}\int_{\mathbb{R}^N}\frac{u(z)}{(|x-z|^2+y^2)^{\frac{N+2s}{2}}}dz, \qquad (-\Delta)^s u(x) = d_{N,s}\int_{\mathbb{R}^N}\frac{u(x)-u(y)}{|x-y|^{N+2s}}dy.$$

The appropriate functional spaces to work with are the Sobolev spaces $\dot{H}^s(\mathbb{R}^N)$ and $X^s(\mathbb{R}^{N+1}_+)$, defined as the completion of $\mathcal{C}_0^\infty(\mathbb{R}^N)$ and $\mathcal{C}_0^\infty(\overline{\mathbb{R}^{N+1}_+})$ respectively, under the norms

$$\|\psi\|^2_{\dot{H}^s} = \int_{\mathbb{R}^N}|(-\Delta)^{\frac{s}{2}}\psi(x)|^2dx, \qquad \|\phi\|^2_{X^s} = \kappa_s\int_{\mathbb{R}^{N+1}_+} y^{1-2s}|\nabla\phi(x,y)|^2dxdy.$$

The extension operator $E_s : \dot{H}^s(\mathbb{R}^N) \to X^s(\mathbb{R}^{N+1}_+)$, $u \mapsto w = E_s[u]$, is an isometry between $\dot{H}^s(\mathbb{R}^N)$ and $X^s(\mathbb{R}^{N+1}_+)$, that is, $\|\varphi\|_{\dot{H}^s} = \|E_s[\varphi]\|_{X^s}$ for all $\varphi \in \dot{H}^s(\mathbb{R}^N)$. Moreover, there exists $C = C(N,s) > 0$ such that (cf. [8]), $\|w(\cdot,0)\|_{L^{2^*_s}} \le C\|w\|_{X^s}$ for all $w \in X^s(\mathbb{R}^{N+1}_+)$.

Along the work we will use the following notation:

- $E := W^{s,2}(\mathbb{R}^N)$, denotes the fractional Sobolev space with scalar product and norm
$$(u \mid v)_j = \int_{\mathbb{R}^N}[(-\Delta)^{\frac{s}{2}}u(-\Delta)^{\frac{s}{2}}v + \lambda_j uv]dx, \quad \|u\|_j^2 = (u \mid u)_j, \;\; j = 1,2.$$
- $\mathbb{E} := E \times E$; the elements in $\mathbb{E}$ will be denoted by $\mathbf{u} = (u_1,u_2)$; as a norm in $\mathbb{E}$ we will take $\|\mathbf{u}\|^2_{\mathbb{E}} = \|u_1\|_1^2 + \|u_2\|_2^2$.
- for $\mathbf{u} \in \mathbb{E}$, $\mathbf{u} \ge \mathbf{0}$ (resp. $\mathbf{u} > \mathbf{0}$) means that $u_j \ge 0$, (resp. $u_j > 0$), for all $j = 1,2$.

For $u \in E$, (resp. $\mathbf{u} \in \mathbb{E}$), we set

$$F(\mathbf{u}) = \frac{1}{2p}\int_{\mathbb{R}^N} \mu_1|u_1|^{2p} + \mu_2|u_2|^{2p}dx, \qquad G(\mathbf{u}) = \frac{1}{p}\int_{\mathbb{R}^N} |u_1|^p|u_2|^p dx,$$

$$I_j(u) = \frac{1}{2}\|u\|_j^2 - \frac{1}{2p}\mu_j \int_{\mathbb{R}^N} |u|^{2p}dx, \qquad \Phi(\mathbf{u}) = \frac{1}{2}\|\mathbf{u}\|_{\mathbb{E}}^2 - F(\mathbf{u}) - \beta\, G(\mathbf{u}).$$

We remark that F and G are well defined because $\frac{p-1}{2p}N < s < 1$ guarantees $2p < 2_s^*$ which in turn implies the continuous Sobolev embedding $E \hookrightarrow L^{2p}(\mathbb{R}^N)$.

Any critical point $\mathbf{u} \in \mathbb{E}$ of Φ gives rise to a solution of (1). If $\mathbf{u} \neq \mathbf{0}$ we say that such a critical point is non-trivial. We also say that a solution $\mathbf{u}$ of (1) is *positive* if $\mathbf{u} > \mathbf{0}$.

Definition 1.1. We say $\mathbf{u}=(u_1, u_2)\in\mathbb{E}$ is a *bound state* to (1) iff it is a critical point of Φ.

Definition 1.2. A positive bound state $\mathbf{u} > \mathbf{0}$ is called a *ground state* of (1) if its energy is minimal among *all the non-trivial bound states*, namely

$$\Phi(\mathbf{u}) = \min\{\boldsymbol{\Phi}(\mathbf{v}) : \mathbf{v} \in \mathbb{E} \setminus \{\mathbf{0}\},\ \boldsymbol{\Phi}'(\mathbf{v}) = \mathbf{0}\}.$$

Ground states are candidates to be orbitally stable for evolution equations (cf. [12]).

2. The Nehari manifold and preliminary results

To find critical points of Φ we will use the so-called Nehari manifold approach. Let us set $\Psi(\mathbf{u}) = (\Phi'(\mathbf{u}) \mid \mathbf{u}) = \|\mathbf{u}\|_{\mathbb{E}}^2 - 2p\, F(\mathbf{u}) - 2p\beta\, G(\mathbf{u})$, then, we define the Nehari manifold as

$$\mathcal{M} = \{\mathbf{u} \in \mathbb{E}_{rad} \setminus \{\mathbf{0}\} : \Psi(\mathbf{u}) = 0\},$$

where *rad* means radial. Plainly, $\mathcal{M}$ contains all the non-trivial critical points of Φ on $\mathbb{E}_{rad}$.

Proposition 2.1. *We have that $\mathbf{u} \in \mathbb{E}$ is a non-trivial critical point of Φ if and only if $\mathbf{u} \in \mathcal{M}$ and is a critical point of Φ constrained to $\mathcal{M}$.*

Consequently, $\mathcal{M}$ is called a *natural constraint* for Φ. The key point of the Nehari manifold approach is that Φ is bounded from below on $\mathcal{M}$ so that one can try to minimize Φ on $\mathcal{M}$.

Concerning the Palais-Smale (PS) condition, for $N = 1$, we have no compact embedding of E into $L^q(\mathbb{R})$ for any $1 < q < 2_s^*$. Nevertheless, we will prove that for a given PS sequence we can find a subsequence its weak limit is a bound state. By the compact embeddings in the radial case, for $1 < N \leq 3$ the PS condition follows by a standard argument.

Lemma 2.2. *Assume $1 < N \leq 3$. Then Φ satisfies the (PS) condition on $\mathcal{M}$: every $\mathbf{u}_n \in \mathcal{M}$ such that $\Phi(\mathbf{u}_n) \to c$ and $\nabla_{\mathcal{M}}\Phi(\mathbf{u}_n) \to 0$ has a strongly convergent subsequence.*

Next, we need some existence results for the decoupled equations that allow us to state the character as critical points of the semi-trivial solutions. To that end, we recall that

$$(-\Delta)^s u + u = |u|^\alpha u \quad \text{in } \mathbb{R}^N, \quad u \in E, \quad u \not\equiv 0, \tag{4}$$

has a unique radial positive solution (cf. [19, 20]) for $0 < \alpha < \frac{4s}{N-2s}$. It is clear that, for any $\beta \in \mathbb{R}$, system (1) has two semi-trivial positive solutions, $\mathbf{u}_1 = (U_1, 0)$ and $\mathbf{u}_2 = (0, U_2)$, where U_j is the unique radial positive solution of

$$(-\Delta)^s u + \lambda_j u = \mu_j |u|^{2p-2} u. \tag{5}$$

Thus, if we set $U_j(x) = \left(\dfrac{\lambda_j}{\mu_j}\right)^{\frac{1}{2p-2}} U(\lambda_j^{\frac{1}{2s}} x)$, $j = 1, 2$, with U the unique positive radial solution of (4), then U_j are solutions of (5). Hence, to find non-trivial solutions, one has to find solutions having *both components* not identically zero.

We are ready to show existence of non-negative solutions of (1) different from $\mathbf{u}_j$, $j = 1, 2$. Let us define $\Lambda = \min\{\gamma_{12}, \gamma_{21}\}$ and $\Lambda' = \max\{\gamma_{12}, \gamma_{21}\}$ where

$$\gamma_{12} = \inf_{\varphi \in E_{rad}\setminus\{0\}} \frac{\|\varphi\|_2^2}{\displaystyle\int_{\mathbb{R}^N} U_1^2 \varphi^2 dx}, \qquad \gamma_{21} = \inf_{\varphi \in E_{rad}\setminus\{0\}} \frac{\|\varphi\|_1^2}{\displaystyle\int_{\mathbb{R}^N} U_2^2 \varphi^2 dx}.$$

The existence of nontrivial nonnegative solutions to (1) relies on the next result.

Proposition 2.3. *The following holds:*

(1) *If $p = 2$, then*
  - (i) *for any $\beta < \Lambda$, the semi-trivial solutions $\mathbf{u}_j$, are strict local minima of $\Phi|_{\mathcal{M}}$.*
  - (ii) *for any $\beta > \Lambda'$, the semi-trivial solutions $\mathbf{u}_j$, $j = 1, 2$, are saddle points of $\Phi|_{\mathcal{M}}$. In particular, we have $\inf_{\mathcal{M}} \Phi < \min\{\Phi(\mathbf{u}_1), \Phi(\mathbf{u}_2)\}$.*

(2) *If $p > 2$, for any $\beta \in \mathbb{R}$, the semi-trivial solutions $\mathbf{u}_j$ are strict local minima of $\Phi|_{\mathcal{M}}$.*

3. Existence Results

By Proposition 2.1, to find a non-trivial solution of (1) it is enough to find a critical point of $\Phi|_{\mathcal{M}}$. This will follow from Proposition 2.3 and the PS condition (Lemma 2.2 if $N = 2, 3$).

Proposition 3.1. *The following holds:*

(1) *If $p = 2$,*
  - (i) *for any $\beta < \Lambda$, the functional Φ has a Mountain-Pass (MP) critical point $\mathbf{u}^*$ on $\mathcal{M}$. Moreover, one has $\Phi(\mathbf{u}^*) > \max\{\Phi(\mathbf{u}_1), \Phi(\mathbf{u}_2)\}$.*
  - (ii) *for any $\beta > \Lambda'$, the functional Φ has a global minimum $\widetilde{\mathbf{u}}$ on $\mathcal{M}$. Moreover, one has $\Phi(\widetilde{\mathbf{u}}) < \min\{\Phi(\mathbf{u}_1), \Phi(\mathbf{u}_2)\}$.*

(2) *If $p > 2$, for any $\beta \in \mathbb{R}$ the functional Φ has a MP critical point $\mathbf{u}^*$ on $\mathcal{M}$. Moreover, one has $\Phi(\mathbf{u}^*) > \max\{\Phi(\mathbf{u}_1), \Phi(\mathbf{u}_2)\}$.*

3.1. **Existence of ground states.**

This result relies on Proposition 3.1–(1)-(ii), providing a MP critical point $\widetilde{\mathbf{u}}$ with $\Phi(\widetilde{\mathbf{u}}) < \min\{\Phi(\mathbf{u}_1), \Phi(\mathbf{u}_2)\}$, which will lead to a ground state.

Theorem 3.2. *If $p = 2$ and $\beta > \Lambda'$, system (1) has a positive radial ground state $\widetilde{\mathbf{u}}$.*

3.2. **Existence of bound states.**

Proposition 3.1–(1)-(i), –(2)-(i) provide MP critical points with energy greater than $\max\{\Phi(\mathbf{u}_1), \Phi(\mathbf{u}_2)\}$ and, hence, will not lead a ground state. The restriction $\beta > 0$ arises naturally in order to apply the strong maximum principle.

Theorem 3.3. *The following holds:*

- (i) *Assuming $p = 2$ and $\beta < \Lambda$, the system (1) has a radial bound state $\mathbf{u}^*$ such that $\mathbf{u}^* \neq \mathbf{u}_j$, $j = 1, 2$. Moreover, if $0 < \beta < \Lambda$, then $\mathbf{u}^* > 0$.*
- (ii) *Assuming $p > 2$ and $\beta \in \mathbb{R}$, the system (1) has a radial bound state $\mathbf{u}^*$ such that $\mathbf{u}^* \neq \mathbf{u}_j$, $j = 1, 2$. Moreover, if $\beta > 0$, then $\mathbf{u}^* > 0$.*
- (iii) *If $p \geq 2$ and $\beta = \varepsilon b$ and $|\varepsilon|$ small enough, then system (1) has a radial bound state $\mathbf{u}_\varepsilon^*$ such that $\mathbf{u}_\varepsilon^* \to \mathbf{z} := (U_1, U_2)$ as $\varepsilon \to 0$. Moreover, if $\beta = \varepsilon b > 0$ then $\mathbf{u}_\varepsilon^* > \mathbf{0}$.*

4. Some results for systems with more than 2 equations

The above arguments allow us to prove weaker existence results for systems with $m \geq 3$ equations, indeed to prove existence of *nonnegative* bound and ground state solutions. However, following [25] and [3], we will prove the existence of *positive* radial ground and bound states respectively. To simplify, we start showing the results for the system

$$\left\{\begin{array}{l}(-\Delta)^s u_1 + \lambda_1 u_1 = \mu_1 |u_1|^{2p-2}u_1 + \beta_{12}|u_2|^p|u_1|^{p-2}u_1 + \beta_{13}|u_3|^p|u_1|^{p-2}u_1,\\ (-\Delta)^s u_2 + \lambda_2 u_2 = \mu_2 |u_2|^{2p-2}u_2 + \beta_{12}|u_1|^p|u_2|^{p-2}u_2 + \beta_{23}|u_3|^p|u_2|^{p-2}u_2,\\ (-\Delta)^s u_3 + \lambda_3 u_3 = \mu_3 |u_3|^{2p-2}u_1 + \beta_{13}|u_1|^p|u_3|^{p-2}u_3 + \beta_{23}|u_2|^p|u_3|^{p-2}u_3.\end{array}\right. \tag{6}$$

We have now three explicit solutions of (6): $\mathbf{u}_1 = (U_1, 0, 0)$, $\mathbf{u}_2 = (0, U_2, 0)$, $\mathbf{u}_3 = (0, 0, U_3)$ with U_j solution of (5). Moreover, there could be solutions $\mathbf{u} = (u_1, u_2, u_3)$ having one component equal to 0. Indeed, if $u_k \equiv 0$, the pair (u_i, u_j), $i, j \neq k$, solves (1) with $\beta = \beta_{ij}$. Then, for any (u_i, u_j) solving (1), the function $\mathbf{u}$ with the remaining component equal to 0 solves (6). We denote by $\mathbf{u}_{ij}$ these specific solutions. As in Proposition 2.3, we have:

(1) If $p = 2$, then
  - (i) the semi-trivial solutions $\mathbf{u}_i$, $i = 1, 2, 3$, are strict local minima of $\Phi|_{\mathcal{M}}$ provided
$$\beta_{ij} < \gamma_{ij} \qquad \forall\, i, j = 1, 2, 3,\ i \neq j. \tag{7}$$
  - (ii) the semi-trivial solutions $\mathbf{u}_i$, $i = 1, 2, 3$, are saddle points of $\Phi|_{\mathcal{M}}$ provided
$$\forall\, i = 1, 2, 3, \quad \exists j \neq i \quad \text{such that} \quad \beta_{ij} > \gamma_{ij}. \tag{8}$$

(2) If $p > 2$ the semi-trivial solutions $\mathbf{u}_i$, $i = 1, 2, 3$, are strict local minima of $\Phi|_{\mathcal{M}}$ for all $\beta_{ij} \in \mathbb{R}$, $i, j = 1, 2, 3$, $i \neq j$.

Therefore, as in Proposition 3.1, we deduce that

(1) If $p = 2$,
  - (i) and (7) holds, the functional Φ has a MP critical point $\mathbf{u}^*$ on $\mathcal{M}$ satisfying
$$\Phi(\mathbf{u}^*) > \max_{i=1,2,3} \Phi(\mathbf{u}_i). \tag{9}$$
  - (ii) and (8) holds, then Φ has a global minimum $\widetilde{\mathbf{u}}$ on $\mathcal{M}$ such that
$$\Phi(\widetilde{\mathbf{u}}) < \min_{i=1,2,3} \Phi(\mathbf{u}_i). \tag{10}$$

(2) If $p > 2$, for any $\beta \in \mathbb{R}$ the functional Φ has a MP critical point $\mathbf{u}^*$ on $\mathcal{M}$ such that
$$\Phi(\mathbf{u}^*) > \max_{i=1,2,3} \Phi(\mathbf{u}_i). \tag{11}$$

As for system (1), one can show that $\mathbf{u}^* \geq 0$, $\widetilde{\mathbf{u}} \geq 0$. Nevertheless, although (10) (resp. (9), (11)) implies that $\widetilde{\mathbf{u}} \neq \mathbf{u}_i$, $i = 1, 2, 3$, (resp. $\mathbf{u}^* \neq \mathbf{u}_i$) it does not implies that $\widetilde{\mathbf{u}}$ is not equal to some $\mathbf{u}_{ij}$ (resp. it does not implies $\mathbf{u}^* \neq \mathbf{u}_{ij}$, for some pair i, j). Summarizing,

Theorem 4.1. *If $p = 2$ and (8) holds, system (6) has a nonegative radial ground state $\widetilde{\mathbf{u}}$.*

Theorem 4.2. *The following holds:*

- (i) *If $p = 2$ and (7) holds, the system (6) has a radial bound state $\mathbf{u}^*$ such that $\mathbf{u}^* \neq \mathbf{u}_i$, $i = 1, 2, 3$. Moreover, if all $\beta_{ij} > 0$, then $\mathbf{u}^* \geq 0$.*
- (ii) *If $p > 2$, system (6) has a radial bound state $\mathbf{u}^*$ such that $\mathbf{u}^* \neq \mathbf{u}_i$, $i = 1, 2, 3$ for all $\beta_{ij} \in \mathbb{R}$, $i, j = 1, 2, 3$, $i \neq j$. Moreover, if all $\beta_{ij} > 0$ then $\mathbf{u}^* \geq 0$.*

We can still extend Theorems 3.2 and 3.3 to system (2) with $m \geq 3$ equations and $\beta_{ij} = \beta_{ji}$. A proof similar to [25, Theorem 2.1] allow us to extend Theorem 3.2 to prove existence of positive ground states for any $p \geq 2$, since this technique does not rely on the character of the semi-trivial solutions as a critical points (in contrast with Theorem 3.3, which strongly relies on which type of critical points the semi-trivial solutions are for $\Phi|_{\mathcal{M}}$, an inherited feature from Proposition 2.3). For $\lambda > 0$ let us define

$$\mathcal{E}(\mathbf{u}) = \frac{\|\mathbf{u}\|^2_{\mathbb{E}^m}}{\left(\displaystyle\sum_{j=1}^m \mu_j \int_{\mathbb{R}^N} |u_j|^{2p} dx + \sum_{\substack{i,j=1\\ i\neq j}}^m \beta_{ij} \int_{\mathbb{R}^N} |u_i|^p |u_j|^p dx\right)^{\frac{1}{p}}} \quad \text{and} \quad \Theta_\lambda = \frac{\displaystyle\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} U|^2 + U^2\, dx}{\displaystyle\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} U|^2 + \lambda U^2\, dx}.$$

Theorem 4.3. *Assume that, for some $\lambda > 0$,*

$$(\mathcal{H}) \qquad \sum_{j=1}^m \mu_j \Theta^p_{\frac{\lambda_j}{\lambda}} + \sum_{\substack{i,j=1\\ i\neq j}}^m \beta_{ij} \left(\Theta_{\frac{\lambda_i}{\lambda}} \Theta_{\frac{\lambda_j}{\lambda}}\right)^{\frac{p}{2}} > m^2 \left\{ \max_{1\leq j\leq m} \mu_j \left(\frac{\lambda}{\lambda_j}\right)^{p - \frac{N}{2s}(p-1)} + \max_{\substack{1\leq i,j\leq m\\ i\neq j}} \beta_{ij} \left(\frac{\lambda^2}{\lambda_i \lambda_j}\right)^{\frac{p}{2} - \frac{N}{2s}\left(\frac{p}{2} - \frac{1}{2}\right)} \right\},$$

then, system (2) has a positive radial ground state $\widetilde{\mathbf{u}}$. Moreover, the ground state $\widetilde{\mathbf{u}}$ is given, up to a Lagrange multiplier, by $\inf_{\mathbf{u}\in\mathbb{E}^m\setminus\{\mathbf{0}\}} \mathcal{E}(\mathbf{u})$.

We prove the existence of a *positive bound state* of system (2) for β_{ij} small enough.

Theorem 4.4. *If $\beta_{jk} = \varepsilon b_{ij}$ for $i, j = 1, 2, \ldots, m$, $i \neq j$ and $|\varepsilon|$ small enough, then system (6) has a radial bound state $\mathbf{u}_\varepsilon$ such that $\mathbf{u}_\varepsilon \to \mathbf{z} := (U_1, U_2, \ldots, U_m)$ as $\varepsilon \to 0$. Moreover, if $\beta_{ij} = \varepsilon b_{ij} > 0$ then $\mathbf{u}_\varepsilon > \mathbf{0}$.*

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