SOME RECENT DEVELOPMENTS ON STRUCTURED DEFORMATIONS

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Abstract. In this talk I will describe the theory of structured deformations introduced by Del Piero and Owen in [1] as well as a generalization to hierarchies of structured deformations introduced in [4]. Moreover, in order to describe the phenomena of yielding, hysteresis, and dissipation which are central aspects of the plastic behavior of solids, a non-local term in the light of [2], can be introduced. Departing from this Mechanical framework, I will describe the variational approach in [3] as well as our recent contributions to the theory in [5] and [6].

Introduction

In classical theories of the mechanics of continua, the macroscopically observable changes in geometry of a continuous body in N-dimensional physical space are identified with smooth, injective mappings g from a reference region $\Omega \subset \mathbb{R}^N$ into $\mathbb{R}^N$, typically with $N \in \{1, 2, 3\}$. The mechanical response at a point x in Ω to such a "classical" deformation g of the body depends upon the nature of the material composing the body. Elastic bodies provide the simplest example of mechanical response: the energy stored per unit volume at x depends only on the $N \times N$ matrix $\nabla g(x)$, the classical gradient at x of the smooth deformation g of Ω.

Among the variety of models of material behavior that describe departures from elastic response, many incorporate the effects of geometrical changes at submacroscopic levels, i.e., changes that are not observable with the naked eye. In this spirit, Del Piero and Owen [1] envisioned a way to account for the effects at the macroscopic level of not only smooth submacroscopic geometrical changes, but also the effects of non-smooth geometrical changes (*disarrangements*) at both submacroscopic and at macroscopic levels. The theory of Del Piero and Owen features objects of the form (g, G), (first-order structured deformations) where $g : \Omega \to \mathbb{R}^d$ is the deformation at the macroscopic level and $G : \Omega \to \mathbb{R}^{d\times N}$ is the deformation gradient at the sub-macroscopic level.

The variational approach to continuum mechanics involves the introduction of an energy functional which associates with any deformation u of a body an energy, whose minimizers are the equilibrium configurations of the body (possibly subject to external loading). The typical energy functional that is considered features a bulk contribution, measuring the deformation (gradient) throughout the whole body, and an interfacial contribution, accounting for the energy needed for fracturing the body. The general form of such an energy, for simple

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deformations $u : \Omega \to \mathbb{R}^d$, is

$$E(u) = \int_\Omega W(\nabla u(x))\,dx + \int_{S_u\cap\Omega} \psi([u](x), \nu_u(x))\,d\mathcal{H}^{N-1}(x), \tag{1}$$

where $W : \mathbb{R}d \times N \to [0, +\infty)$ is the (possibly non-linear and non-convex) bulk energy density and $\psi : \mathbb{R}^d \times \mathcal{S}^{N-1} \to [0, +\infty)$ is the interfacial energy density. The singular set S_u is the discontinuity set of the deformation u and $\nu_u(x)$ is the unit normal vector to S_u at x. The explicit dependence of ψ on the normal ν_u models the presence of anisotropies in the material.

Energy (1) contains two very well known and studied models in mechanics: by taking $W(\xi) = \frac{1}{2}|\xi|^2$ and $\psi(\lambda, \nu) \equiv 0$, one has the classical Dirichlet energy for linear elasticity; by taking $W(\xi) = \frac{1}{2}|\xi|^2$ and $\psi(\lambda, \nu) = 1$, one has the Griffiths model for fracture mechanics: indeed, the second term reduces to $\mathcal{H}^{N-1}(S_u)$, measuring the length or the area (for $N = 2$ and 3, respectively) of the crack.

Assigning an energy to a structured deformation (g, G) is not a straightforward task. The proposal in [3] is to give the structured deformation (g, G) the energy of the most economical approximation, namely, for every (g, G), we define

$$I(g, G) \coloneqq \inf \Big\{ \liminf_{n\to\infty} E(u_n) : u_n \rightsquigarrow (g, G) \Big\}, \tag{2}$$

where $\rightsquigarrow$ indicates a suitable sense of convergence yet to be specified: $u_n \to g$, $\nabla u_n \to G$. In particular, it is evident that the process described is a relaxation process; therefore a suitable function space for structured deformations must be introduced and a suitable notion of convergence must be specified.

In [5], in order to describe the phenomena of yielding, hysteresis, and dissipation, multi-scale geometrical changes are described via structured deformations (g, G) and the non-local energetic response at a point $x \in \Omega$ via a function Ψ of the weighted averages of the jumps $[u_n](y)$ of admissible deformations u_n at points y within a distance r of x. These weighted averages account for non-linear contributions of $[u_n]$. This averaging contribution is coupled with the model of Choksi-Fonseca in [3].

In [4], Deseri and Owen extended the theory of [1] to *hierarchies* of structured deformations in order to include the effects of disarrangements at more than one sub-macroscopic level. This extension is based on the fact that many natural and man-made materials exhibit different levels of disarrangements. Muscles, cartilage, bone, plants and biomedical materials are just some of the materials whose mechanical behavior can be addressed within this generalized field theory. The energetic relaxation to first order structured deformations of [3] is extended to hierarchies of structured deformations in [6].

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[6] Ana C. Barroso, J. Matias, M. Morandotti, D. R. Owen and E. Zappale The variational modeling of hierarchies of structured deformations work in progress

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