CALCULUS OF VARIATIONS WITH FRACTIONAL DERIVATIVES

DELFIM F. M. TORRES

Abstract. A Calculus of Variations can be formulated to include derivatives of fractional (non-integer) order. Lagrangians with fractional derivatives lead directly to equations of motion with nonconservative forces, such as friction, circumventing Bauer's corollary that "The equations of motion of a dissipative linear dynamical system with constant coefficients are not given by a variational principle." We continue the development of the fractional-derivative calculus of variations, providing a new Lagrangian for which the respective Euler–Lagrange equations coincide with the equations of motion of a dissipative system.

Fractional differentiation means "differentiation of arbitrary order". Its origin goes back more than 300 years, when in 1695 L'Hopital asked Leibniz the meaning of $\frac{d^n y}{dx^n}$ for $n = \frac{1}{2}$. After that, many famous mathematicians, like Fourier, Abel, Liouville, and Riemann, among others, contributed to the development of Fractional Calculus [11].

In 1931, Bauer proved that it is impossible to use a variational principle to derive a single linear dissipative equation of motion with constant coefficients [4]. Bauer's result expresses the well-known belief that there is no direct method of applying variational principles to nonconservative systems, which are characterized by friction or other dissipative processes. It turns out that fractional derivatives provide an elegant solution to the problem. Indeed, the proof of Bauer's theorem relies on the tacit assumption that all derivatives are of integer order. If a Lagrangian is constructed using fractional derivatives, then the resulting equation of motion can be nonconservative. This was first proved by Riewe in 1996/97 [9, 10], giving rise to the beginning of the *Fractional Calculus of Variations* (FCV). For those interested on the FCV we refer to the books [1, 2, 6, 7] and the survey papers [3, 8].

In 2015, Caputo and Torres introduced a duality theory between the left and right fractional operators [5]. Roughly speaking, they have shown that the right fractional operators of a function are the dual of the left operators of the dual function or, equivalently, the left fractional derivative/integral of a function is the dual of the right fractional derivative/integral of the dual function. Here we use such duality theory to provide a new perspective to the FCV and obtain a new Lagrangian for which the respective Euler–Lagrange equations coincide with the equations of motion of a dissipative system. In contrast with the example of Riewe, and others available in the literature, our quadratic Lagrangian for the linear friction problem is a a real valued Lagrangian involving let-hand side fractional derivatives only.

References

[1] R. Almeida, S. Pooseh and D. F. M. Torres, *Computational methods in the fractional calculus of variations*, Imperial College Press, London, 2015. https://doi.org/10.1142/p991

The author has been partially supported by FCT and CIDMA, project UIDB/04106/2020. Correspondence to delfim@ua.pt.

[2] R. Almeida, D. Tavares and D. F. M. Torres, *The variable-order fractional calculus of variations*, SpringerBriefs in Applied Sciences and Technology, Springer, Cham, 2019. https://doi.org/10.1007/978-3-319-94006-9

[3] R. Almeida and D. F. M. Torres, A survey on fractional variational calculus, in *Handbook of fractional calculus with applications. Vol. 1*, 347–360, De Gruyter, Berlin, 2019. https://doi.org/10.1515/9783110571622-014

[4] P. S. Bauer, Dissipative Dynamical Systems. I. Proc. Natl. Acad. Sci. **17** (1931), no. 5, 311–314. https://doi.org/10.1073/pnas.17.5.311

[5] M. C. Caputo and D. F. M. Torres, Duality for the left and right fractional derivatives, Signal Process. 107 (2015), 265–271. https://doi.org/10.1016/j.sigpro.2014.09.026

[6] A. B. Malinowska, T. Odzijewicz and D. F. M. Torres, *Advanced methods in the fractional calculus of variations*, SpringerBriefs in Applied Sciences and Technology, Springer, Cham, 2015. https://doi.org/10.1007/978-3-319-14756-7

[7] A. B. Malinowska and D. F. M. Torres, *Introduction to the fractional calculus of variations*, Imperial College Press, London, 2012. https://doi.org/10.1142/p871

[8] T. Odzijewicz and D. F. M. Torres, The generalized fractional calculus of variations, Southeast Asian Bull. Math. **38** (2014), no. 1, 93–117.

[9] F. Riewe, Nonconservative Lagrangian and Hamiltonian mechanics, Phys. Rev. E (3) **53** (2) (1996) 1890–1899. https://doi.org/10.1103/PhysRevE.53.1890

[10] F. Riewe, Mechanics with fractional derivatives, Phys. Rev. E (3) **55** (3) (1997) 3581–3592. https://doi.org/10.1103/PhysRevE.55.3581

[11] S. G. Samko, A. A. Kilbas and O. I. Marichev, *Fractional integrals and derivatives*, Gordon and Breach Science Publishers, Yverdon, 1993.

Delfim F. M. Torres; R&D Unit CIDMA, Department of Mathematics, University of Aveiro
Email address: delfim@ua.pt