AUTOMATED REASONING IN GEOMETRY BY MEANS OF COMPUTER ALGEBRA: REAL AND COMPLEX ISSUES

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Abstract. Our aim is to report the current development in GeoGebra of geometric automated reasoning tools by means of computational algebraic geometry algorithms. We describe our approach by providing algorithms, using computer algebra methods, for confirming/refuting theorems (or for discovery new ones) over some given geometric configuration. Then we attempt to address some pending issues concerning the automatic reasoning in the real case, but with some limitations. Thus, we do not consider the inclusion of inequations in the formulation of the statement, rather we focus on finding some framework over the reals that follows closely the complex case, to profit from a possible real/complex interaction, as the complexity of the implemented complex geometry algorithms is usually much better than that of the algorithms we should use over the reals. We will provide a first detailed proposal to address these issues, exemplified through a particularly motivating example (Clough's conjecture).

Extended abstract

Roughly speaking, GeoGebra's automated reasoning tools for elementary geometry implement the algorithmic approach of [9] and [7]. It involves translating hypotheses H and thesis T into sets of equations, and checking the inclusion $V(H) \subset V(T)$ of the corresponding zero-sets over an algebraically closed field, such as the complex field.

But this straightforward formulation is hardly useful in practice, as $V(H)$ often includes unexpected instances where the thesis does not hold, e.g. related to limit cases such as lines defined by two points, when these two points coincide, or to lines parallel to a given one passing through a point, when this point happens to lie in the given line, etc. Thus, the theoretical framework in [9] included a more sophisticated proposal, involving the algorithmic detection and marginalization (in some sense) of these degenerate instances, by highlighting a distinguished set of geometrically meaningful, free variables, ruling $V(H)$. Finally, the initial inclusion test $V(H) \subset V(T)$ is reconsidered and conducted by means of algorithms concerning the elimination of the ideals $(H, T * t - 1)$ and (H, T) over the chosen set of relevant variables, leading to the concepts of "generally true", "generally false" or "true on parts" when the output of the corresponding elimination is different or equal to zero.

The quite successful performance of this approach (see [1, 8]) relies, among other reasons, on the choice of the complex algebraic geometry context –and its highly developed algorithms–for developing the theoretical framework. It has three counterparts: one, it implies reasoning (stating its truth or failure) over all complex instances of elementary geometry statements; two, a lack of precision on the choice of signs, when lenghts, angle

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bisectors, etc. are involved, as the chosen theoretical framework avoids including inequations in the description of $V(H)$; likewise, no statements concerning inequalities can be approached in this context.

The first drawback seems, in practical terms, not very relevant, as it seems most statements being true over the reals are also true over the complexes (see [2] for a detailed comment and many references to this long-time standing complicated relation).

The second issue has been subject to different analysis and proposals, such as the use of MEP (Minimal Euclidean Polynomials) (see [6]) to keep working in the complex setting, but with some special consideration to the sign-choice problem.

The third problem, about handling inequalities, has been partially addressed in some cases through the consideration of the associated equality and of non-automatic reasoning analysis of the output to detect the impact of signs, for example, in [3].

Of course, it is obvious that deciding, algorithmically, some statement $H \to T$, over the reals (even including inequations in its formulation), can be addressed through real elimination of quantifiers, as described in [2], where the current quite successful approach to automated reasoning implemented in GeoGebra (see [8]) is summarily analyzed from the theorem-proving-over-the- reals issue, highlighting, as main difficulties to adapt quantifier elimination algorithms for the real context, the need to reconsider the concepts of component, dimension, degeneracy, of the Hypotheses variety, as well as the elimination algorithms, mentioning succintly that they should be replaced by some projection tools.

In this talk we attempt to address some of this pending issues concerning the automatic reasoning in the real case, but with some limitations, motivated by the complexity of the implemented complex geometry algorithms which is usually much better than that of the algorithms we should use over the reals. Thus, we do not consider the inclusion of inequations in the formulation of the statement, rather we focus on finding some framework over the reals that follows closely the complex case, to take advantage of a possible real/complex interaction that allows us to describe in an analogous way the geometrical situation posed.

The goal is to provide a first detailed proposal to address the above mentioned issues, exemplified through a particularly motivating example, Clough's conjecture. This conjecture is about equilateral triangles and was introduced by De Villiers in [4, 5].

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