Zeros of derivatives of real entire functions

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An entire function is called real if it takes real values on the real axis. We are interested in the number of nonreal zeros of the derivatives of a real entire function. This subject a long history, going back to the work of Fourier on the reality of the zeros of Bessel functions.

The Laguerre-Pólya class consists of all real entire functions which are locally uniform limits of real polynomials with real zeros. It follows that if f is in the Laguerre-Pólya class, then so are all derivatives of f. In particular, the zeros of the derivatives of f are also all real. On the other hand, if f is a real entire function with real zeros, but not in the Laguerre-Pólya class, one would expect that some derivative of f has non-real zeros. In this context, the following conjectures were made by Wiman (1911) and Pólya (1943):

Conjecture 1. f'' has non-real zeros.

Conjecture 2. The number of non-real zeros of $f^{(n)}$ tends to infinity with n.

Conjecture 1 was confirmed in 1960 by Levin and Ostrovskii for functions of sufficiently fast growth, and in 1989 by Sheil-Small for functions of finite order. The remaining case was dealt with by Bergweiler, Eremenko and Langley in 2003, thus completing the proof of the conjecture.

Conjecture 2 was proved by Hellerstein and Yang in 1971 under the assumptions made by Levin and Ostrovskii. In 2005 it was then proved by Langley for all functions of infinite order. The remaining case of finite order was recently settled by Bergweiler and Eremenko. The techniques used here are very different from the ones used for the infinite order case or for Conjecture 1.

The talk will give further background to these conjectures, describe some of the previous results that have been obtained and the techniques that have been used, and sketch the main ideas in the proof of the second conjecture.