Invariant subspaces of Hardy spaces of a domain and of a Riemann surface

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Let R be a (bounded) Riemann surface and $z : R \to \mathbb{C}$ a bounded analytic function. We discuss the problem of describing all invariant subspaces of the operator of multiplication by z, acting on the Hardy space $H^p(R)$, $1 \leq p < \infty$. The problem is interesting even when R is a bounded subdomain of $\mathbb{C}$ with analytic boundary and z is the identity map.

If R is a simply connected domain, then, due to the Beurling Theorem, all invariant subspaces $E \subset H^p(R)$ have the form $E = \theta H^p(R)$, where $\theta \in H^\infty(R)$ is an inner function. To obtain the answer for other cases, basically two different approaches are known. The first one (Hitt, Sarason, Hartmann, Seip and others) involves extremal functions and works better for the Hilbert space case, where it gives the most definite results. The description of invariant subspaces of H^2 of a multiply connected domain, which was obtained in this way by Hitt and Sarason for the case of the annulus (1988) and the speaker (1989), involves the so-called *nearly invariant subspaces* M of $H^2(\mathbb{D})$, which have the following property:

$$M \subset H^2(\mathbb{D}),\ f \in M, f(0) = 0 \Rightarrow \frac{f(z)}{z} \in M.$$

The second approach, which goes back to Volberg and Solomyak, involves the A-integral of functions in weak L^1 spaces, introduced by Alexei Alexandrov, and a certain "glueing" lemma for functions in $H^{1,\mathrm{weak}}$. If R is a Riemann surface that satisfies a certain geometric condition, then one can write down a formula for an arbitrary invariant subspace of $H^p(R)$.

In this talk, we will describe both approaches. We will also mention recent results by Aleman, Feldman and Ross on invariant subspaces in slit discs and by Alexander Kiselev and the speaker concerning invariant subspaces of the vector Hardy space $H^2(R, \mathbb{C}^n)$. If time permits, we also will give some motivations coming from Operator Theory.