On a Problem of Beurling

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Abstract

Let Ω be an arbitrary bounded simply connected domain in the complex plane, and let w be a positive continuous function on Ω such that $w(z) \to 0$ as $z \to \partial\Omega$. Denote by $C_w(\Omega)$ the Banach space of all complex-valued functions f for which the product $f(z)w(z)$ is continuous on the closure of Ω and vanishes on $\partial\Omega$, the norm being defined by

$$\|f\| = \sup_{\Omega} |f|w.$$

Evidently, the collection of functions

$$A_w(\Omega) = \{f \in C_w(\Omega) : f \text{ is analytic in } \Omega\}$$

is a closed subspace of $C_w(\Omega)$. The problem raised by Beurling is to determine whether or not the polynomials are dense in $A_w(\Omega)$. It is my intention to present a solution to this problem which is analogous to Mergeljan's solution to the classical Bernstein problem for weighted polynomial approximation on the real line, and which applies to the most general regions. In the process I will indicate the manner in which these ideas can be used to strengthen an earlier theorem of Beurling which can be found in his collected works. I will only indicate briefly some of the more technical results that are needed, and which include:

1. A strengthened version of Thomson's theorem on mean-square polynomial approximation.

2. Tolsa's theorem on the semi-additivity of analytic capacity.

3. Beurling's results on quasianalyticity and general distributions.