THE UNBOUNDED PRODUCT OF NORMAL OPERATORS

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The product of two bounded normal operators need not remain normal. Many simple counterexamples can be constructed with ease. It is, however, well known that the product of two commuting bounded normal operators stays normal. This is a direct application of the bounded version of the well-known Fuglede theorem. I have recently been interested in extending this result to unbounded operators. The naive generalization to the case of one bounded and the other unbounded normal operators (with even stronger assumptions) does not lead to the desired result. This will be illustrated by a counterexample during the talk. Then, positive results as well as some interesting counterexamples will be given in a chronological order as they were obtained by the author. The important tools to establish these results are:

- The unbounded version of the Fuglede theorem.
- The self-adjointness of the unbounded AA^* and A^*A for closed A.
- The fact that unbounded self-adjoint operators are maximally symmetric.
- The relation $(BA)^* = A^*B^*$ which holds if A is *invertible* and B is ***unbounded***.
- The closedness of AB and BA under given conditions.

The main cases treated here are:

(1) The normality of AB and that of BA with A unitary and B unbounded and normal.
(2) Under which conditions do we have AB normal iff BA is normal (where one operator is unbounded)?
(3) The normality of AB ad that of BA for a bounded not necessarily unitary A.
(4) The normality of AB where both A and B are unbounded and normal.

References

1. M. H. Mortad, On the Closedness, the Self-adjointness and the Normality of the product of two unbounded operators", *Demosntratio Math.*, **45/1** (2012) 161-167.
2. M. H. Mortad, K. Gustafson, Kh. Madani, More on the Normality of the Unbounded Product of Two Normal Operators, *(submitted)*. arXiv: 1202.6142.
3. M. H. Mortad, The Sum of Two Unbounded Linear Operators: Closedness, Self-adjointness, Normality and Much More, *(submitted)*. arXiv:1203.2545
4. M. H. Mortad, Products of Unbounded Normal Operators, *(submitted)*. arXiv:1202.6143.

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