An $\mathcal{H}_\infty^-$-calculus motivated from system theory

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In system theory it is well-known that the Toeplitz operator $M_g : L^2(0,\infty) \to L^2(0,\infty)$ with symbol g, which is bounded analytic on the left half plane $\mathbb{C}^-$, maps exponentials to exponentials,

$$M_g(e^{at}) = g(a)e^{at} \tag{1}$$

for fixed $a < 0$. Obviously, $g \mapsto g(a)$ is a homomorphism from $\mathcal{H}_\infty^-$, the Banach algebra of bounded analytic functions on $\mathbb{C}^-$, to $\mathbb{C}$. Our idea is to replace the exponential by the strongly continuous semigroup $e^{At} = T(t)$ on the Banach space X. In fact, we show that the formally defined function

$$y(t) = M_g(T(.)x_0)(t)$$

can be seen as the *output* of the linear system

$$\begin{aligned} \dot{x}(t) &= Ax(t), \qquad x(0) = x_0 \\ y(t) &= C_g x(t) \end{aligned}$$

for some (unbounded) operator C_g. Thus, formally $y(t) = C_g T(t)x_0$. This means that C_g takes the role of $g(a)$ in (1). Hence, the task is to find C_g given the *output mapping* $x_0 \mapsto y(t)$. Incorporating the notion of *admissibility*, this can be done uniquely, see [2]. This construction yields an (unbounded) functional calculus. Moreover, $g(A)$ is bounded from $X_1 = (D(A), \|.\|_A)$ to X and weakly admissible, i.e.

$$\int_0^\infty |\langle y, g(A)T(t)x\rangle_{X',X}|^2 \, dt \le K_A \|g\|_\infty^2 \|x\|^2 \|y\|_{X'}^2, \qquad x \in D(A), y \in X'.$$

Furthermore, if X is a Hilbert space, for every positive time t, $g(A)T(t)$ can be extended to a bounded operator with norm less than $\gamma_A t^{-1/2}$, with γ_A independent of t. Finally, we give sufficient conditions for a bounded calculus, such as *exact observability* for Hilbert spaces (or *exact observability by direction* in the general case), [1, 3].

References

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