Optimal divisions of a convex body

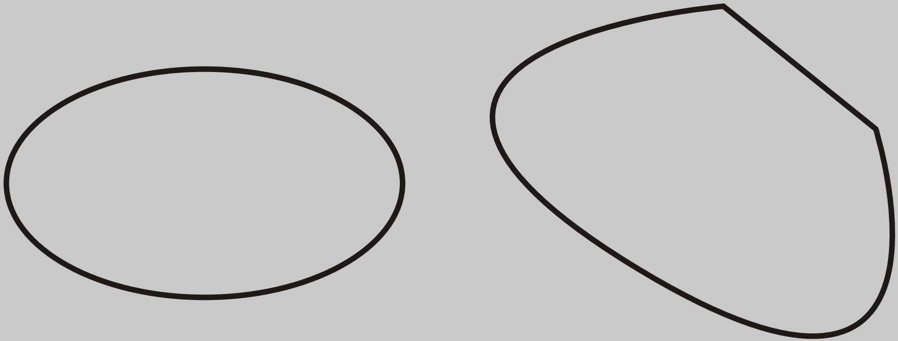
Antonio Cañete (Universidad de Sevilla)

Joint work with Isabel Fernández y Alberto Márquez (Universidad de Sevilla)

Motivation:

$C \subset \mathbb{R}^d$ convex body

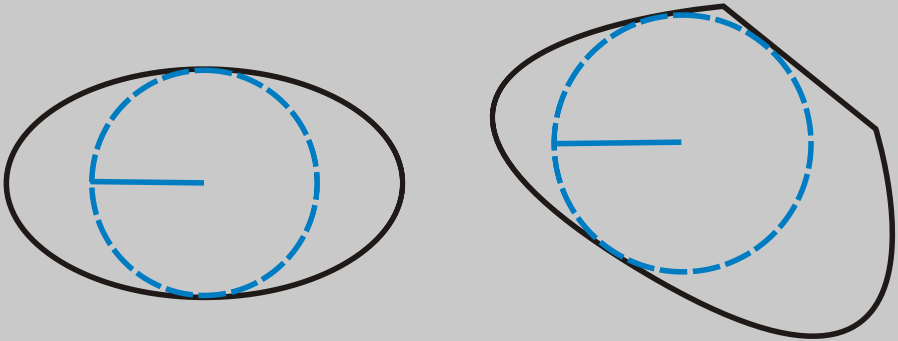
The inradius of C is the radius of the largest ball contained in C



Motivation:

$C \subset \mathbb{R}^d$ convex body

The inradius of C is the radius of the largest ball contained in C



Inradii of these two convex bodies

Motivation:

$C \subset \mathbb{R}^d$ convex body

The inradius of C is the radius of the largest ball contained in C

- Conway's fried potato problem:

Dividing C into n subsets (by using $n-1$ successive hyperplane cuts) minimizing the largest of the inradii of the subsets

Motivation:

$C \subset \mathbb{R}^d$ convex body

The inradius of C is the radius of the largest ball contained in C

- Conway's fried potato problem:

Dividing C into n subsets (by using $n-1$ successive hyperplane cuts) minimizing the largest of the inradii of the subsets

$\rightsquigarrow$ This is a min-Max type problem for the inradius

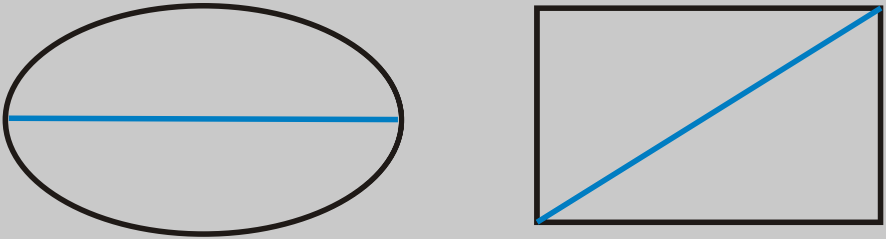
---

A. Bezdek, K. Bezdek, Bull. London Math. Soc., 1995

Motivation:

$C \subset \mathbb{R}^d$ convex body

The diameter of C is the largest distance between two points in C



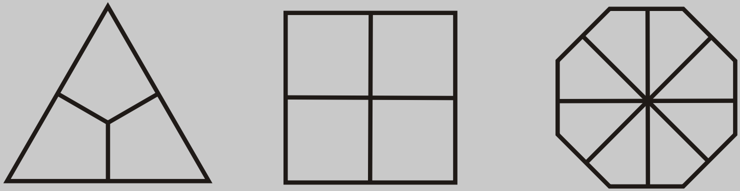
Diameter of an ellipse and a rectangle

Motivation:

$C \subset \mathbb{R}^d$ convex body

The diameter of C is the largest distance between two points in C

- Dividing C into n subsets minimizing the largest of the diameters of the subsets



C. Peri, S. Segura, A. Cañete, U. Schnell, B. González,...

Motivation:

$C \subset \mathbb{R}^d$ convex body

The diameter of C is the largest distance between two points in C

• Dividing C into n subsets minimizing the largest of the diameters of the subsets

$\rightsquigarrow$ This is another min-Max type problem for the diameter

Our project:

$C \subset \mathbb{R}^d$ convex body

- Divisions of C into n subsets (with non-empty interior),

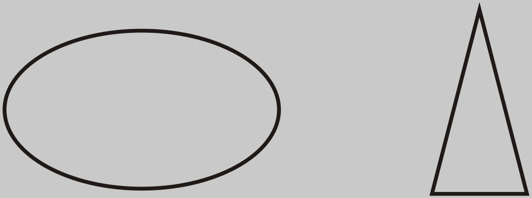
determined by $n-1$ successive hyperplane cuts

Our project:

$C \subset \mathbb{R}^d$ convex body

- Divisions of C into n subsets (with non-empty interior), determined by $n-1$ successive hyperplane cuts

- Classical geometric functionals:
Inradius, Diameter, Width



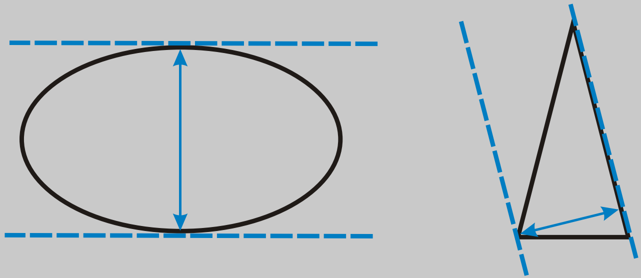
Our project:

$C \subset \mathbb{R}^d$ convex body

- Divisions of C into n subsets (with non-empty interior), determined by $n-1$ successive hyperplane cuts

- Classical geometric functionals:

Inradius, Diameter, Width



Width of an ellipse and a triangle

Our project:

$C \subset \mathbb{R}^d$ convex body

- Divisions of C into n subsets (with non-empty interior),

determined by $n-1$ successive hyperplane cuts

- Classical geometric functionals:

Inradius, Diameter, Width

- min-Max type problem

Our project:

$C \subset \mathbb{R}^d$ convex body

- Divisions of C into n subsets (with non-empty interior), determined by $n-1$ successive hyperplane cuts

- Classical geometric functionals: Inradius, Diameter, Width

- min-Max and Max-min type problem

Our project:

$C \subset \mathbb{R}^d$ convex body

$\rightsquigarrow$ Optimal divisions of C for the min-Max (and Max-min) type problem for each geometric functional

Our project:

$C \subset \mathbb{R}^d$ convex body

$\rightsquigarrow$ Optimal divisions of C for the min-Max (and Max-min) type problem for each geometric functional

What can we study?

- Existence of solution

- Uniqueness of solutions

- Balanced solutions. Are all the solutions balanced?

- Optimal values (or bounds for them)

Existence of optimal divisions:

We have existence in all cases, except for the Max-min type problem for the diameter

- $\rightsquigarrow$ Blaschke Selection Theorem
- $\rightsquigarrow$ When optimal value is known

Existence of optimal divisions:

- Example: min-Max type problem for the diameter

$C \subset \mathbb{R}^d$ convex body, $n \in \mathbb{N}$

optimal value: $D_n(C) = \inf\{D(P) : P$ división de $C\}$

$\underset{\substack{\downarrow \\ C_1^k,\dots,C_n^k}}{\{P_k\}}$ sequence such that $\{\underset{\substack{\| \\ D(C_1^k)}}{D(P_k)}\} \to D_n(C)$

Blaschke Theorem: $\{C_1^k\} \to C_1^\infty, \dots, \{C_n^k\} \to C_n^\infty$

$\{C_1^\infty, \dots, C_n^\infty\} \rightsquigarrow P^\infty$ optimal division of C since

$$D(P^\infty) = D(C_1^\infty) = \lim_k D(C_1^k) = D_n(C)$$

Existence of optimal divisions:

- Example: min-Max type problem for the width

$C \subset \mathbb{R}^d$ convex body, $n \in \mathbb{N}$

P division of C into subsets $C_1, \dots, C_n$

Call $w(P) = \max\{w(C_1), \dots, w(C_n)\}$

Bang's Lemma: $w(C) \leq \sum_{i=1}^{n} w(C_i) \leq n\, w(P)$

$$\Rightarrow w_n(C) \geq w(C)/n$$

T. Bang, Proc. Amer. Math. Soc., 1951

Existence of optimal divisions:

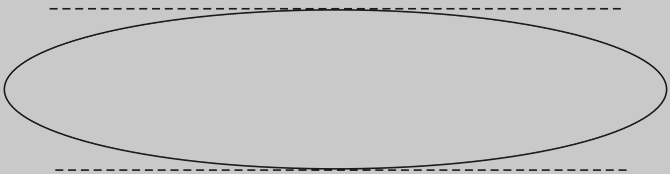
- Example: min-Max type problem for the width

$C \subset \mathbb{R}^d$ convex body, $n \in \mathbb{N}$

P division of C into subsets $C_1, \ldots, C_n$

Bang's Lemma: $w(C) \leq \sum_{i=1}^{n} w(C_i) \leq n\, w(P)$

$$\Rightarrow w_n(C) \geq w(C)/n$$



Existence of optimal divisions:

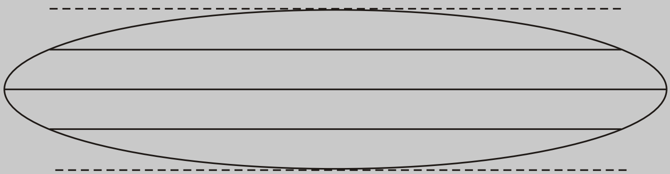
- Example: min-Max type problem for the width

$C \subset \mathbb{R}^d$ convex body, $n \in \mathbb{N}$

P division of C into subsets $C_1, \ldots, C_n$

Bang's Lemma: $w(C) \leq \sum_{i=1}^{n} w(C_i) \leq n\, w(P)$

$$\Rightarrow w_n(C) \geq w(C)/n$$



Existence of optimal divisions:

- Existence is not assured:

Max-min type problem for the diameter

- The optimal value is $D(C)$

Existence of optimal divisions:

- Existence is not assured:

Max-min type problem for the diameter

- The optimal value is $D(C) \Rightarrow$ In an optimal division, the diameter of all the subsets must be $D(C)$

Existence of optimal divisions:

- Existence is not assured:

Max-min type problem for the diameter

- The optimal value is $D(C) \Rightarrow$ In an optimal division, the diameter of all the subsets must be $D(C)$

- If $d \geq 3$, there always exists an optimal division (take $n-1$ hyperplanes containing a diameter segment)

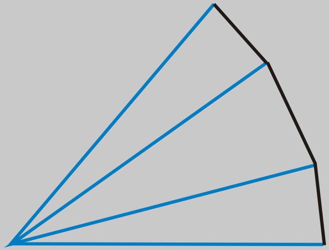
Existence of optimal divisions:

- Existence is not assured:

Max-min type problem for the diameter

- The optimal value is $D(C) \Rightarrow$ In an optimal division, the diameter of all the subsets must be $D(C)$

- If $d = 2$, an optimal division exists if and only if $n \leq 2I + F - \delta_C$



$I = 2,\ F = 2,\ \delta_C = 0$

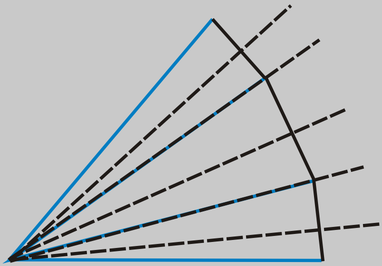
Existence of optimal divisions:

- Existence is not assured:

Max-min type problem for the diameter

- The optimal value is $D(C) \Rightarrow$ In an optimal division, the diameter of all the subsets must be $D(C)$

- If $d = 2$, an optimal division exists if and only if $n \leq 2I + F - \delta_C$



Optimal division for $n = 6$ subsets

Uniqueness of optimal divisions:

$\rightsquigarrow$ There is no uniqueness for these problems

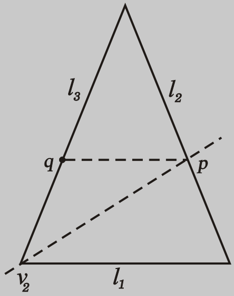
Uniqueness of optimal divisions:

$\rightsquigarrow$ There is no uniqueness for these problems

- Max-min type problem for the diameter: Freedom
- min-Max type problem for the diameter:

Any division of a ball for $n = 2$ is optimal

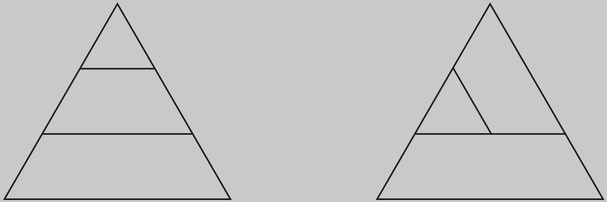
- Max-min type problem for the width:



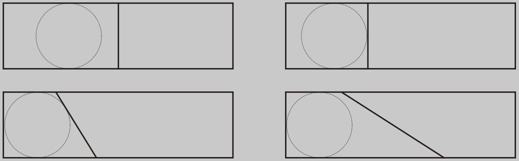
Uniqueness of optimal divisions:

$\rightsquigarrow$ There is no uniqueness for these problems

- min-Max type problem for the width and for the inradius:



- Max-min type problem for the inradius:



Balanced optimal divisions:

A division is balanced if all the subsets have the same value for the considered functional

Balanced optimal divisions:

A division is balanced if all the subsets have the same value for the considered functional

We always have balanced optimal divisions

- By induction

- By using the optimal value

Balanced optimal divisions:

We always have balanced optimal divisions

- min-Max type problem for the width:

All optimal divisions are balanced (Bang's Lemma)

- Max-min type problem for the diameter:

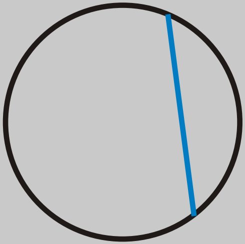
All optimal divisions are balanced (the optimal value is $D(C)$)

Balanced optimal divisions:

We always have balanced optimal divisions

- min-Max type problem for the diameter:

Not all optimal divisions are balanced



Non-balanced optimal división of the disk for $n = 2$

Optimal values:

Optimal values:

- Max-min type problem for the diameter:

  The optimal value is $D(C)$. It may be not attained ($d = 2$)

- min-Max type problem for the width:

  The optimal value is $\dfrac{1}{n} w(C)$

- min-Max type problem for the inradius (Conway's fried potato problem):

  The optimal value is $\rho > 0$ such that $w(C^\rho) = 2n\rho$

Optimal values:

- min-Max type problem for the diameter:

$$\frac{1}{n} D(C) < \text{optimal value} \leq C_0$$

- Max-min type problem for the width:

$$\frac{1}{n} w(C) \leq \text{optimal value} \leq \min\left\{ w(C), \frac{D(C)}{2} \right\}$$

Other questions:

$\rightsquigarrow$ Algorithms leading to the optimal value (and to an optimal division) for a convex polygon

- min-Max type problem for the inradius:

  Quadratic algorithm (medial axis)

- Max-min type problem for the diameter:

  Linear algorithm (rotating calipers)

- min-Max type problem for the width:

  Lineal algorithm

Other questions:

- Max-Max and min-min type problems
- Circumradius and other functionals
- General divisions

Optimal divisions of a convex body

Antonio Cañete (Universidad de Sevilla)

Joint work with Isabel Fernández y Alberto Márquez (Universidad de Sevilla), partially supported by MICINN project PID2020-118180GB-I00

GOBIERNO DE ESPAÑA MINISTERIO DE CIENCIA E INNOVACIÓN