On the volume of Minkowski sum of convex sets.

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Université Gustave Eiffel

Recent (posted the 3rd and 5th of June 2022) work in collaboration with
Mokshay Madiman, Mathieu Meyer and Artem Zvavitch:
F.-Madiman-Zvavitch: Sumset estimates in convex geometry. arXiv:2206.01565.
FMMZ: On the volume of the Minkowski sum of zonoids. arXiv:2206.02123.

geOmetric anaLysis & convExity
Universidad Sevilla, 2022-06-22

A nice conjecture of Thomas Courtade 2018

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Conjecture (Courtade 2018)

Let B, C be compact convex sets in $\mathbb{R}^n$. Is it true that

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More generally one may ask for which compact convex set A, the following inequality holds for every compact convex sets B, C in $\mathbb{R}^n$:

$$(|B||C|)^{1/n} + (|A||A + B + C|)^{1/n} \leq (|A + B||A + C|)^{1/n}?$$

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Analogy between EPI and Brunn-Minkowski

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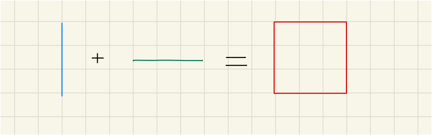

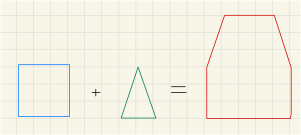

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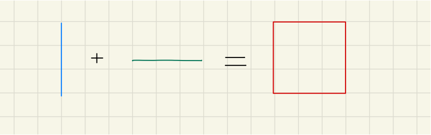

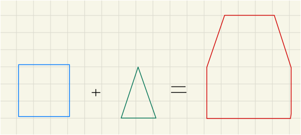

2) Entropy Power Inequality.
Let X and Y be two independent random vectors in $\mathbb{R}^n$. Then

$$N(X+Y) \geq N(X) + N(Y),$$

where for $X \sim f$, $N(X) = \frac{1}{2\pi e} e^{\frac{2}{n}h(X)}$ and $h(X) = -\int f \log(f)$.

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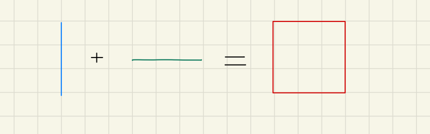

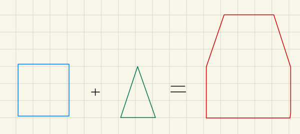

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3) Brunn-Minkowski inequality for matrices.
Let A and B be two non negative symmetric matrices in $\mathcal{M}_n(\mathbb{R})$. Then

$$\det(A+B)^{\frac{1}{n}} \geq \det(A)^{\frac{1}{n}} + \det(B)^{\frac{1}{n}}.$$

Proof: if X Gaussian with covariance matrix A: $N(X) = \det(A)^{\frac{1}{n}}$.

Plan

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Theorem (F.-Marsiglietti (2014))

The conjecture holds true

- *in dimension 1*
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Question: Is it true for $t \geq t(A)$ for any compact A? For example for $t \geq \mathrm{diam(A)}$?

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Monotonicity of entropy: Artstein-Ball-Barthe-Naor (2004)
Let $X_1, \dots, X_m, \dots$ be i.i.d. random vectors then

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Remark: for A convex, one has $A(m) = A$ for any m so the result holds, but is not interesting. So we look at sets A which are compact and non-convex!

Volume of Minkowski averages

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Let A be a compact set in $\mathbb{R}^n$. Then $m \mapsto \mathrm{vol}_n(A(m))$

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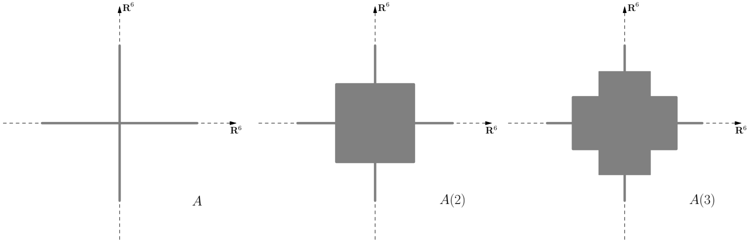

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Theorem (F., Lángi, Zvavitch (2022))

Let A be a starshaped compact set in $\mathbb{R}^n$. Then $\mathrm{vol}_n(A(m)) \leq \mathrm{vol}_n(A(m+1))$ for $m \geq (n-1)(n-2)$, thus yes for $n \leq 3$.

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Artstein-Avidan-Florentin-Ostrover (2014): counterexample for (M).

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But from Giannopoulos-Hartzoulaki-Paouris '03 the following inequality is sharp

$$|A||P_{u,v}A| \leq \frac{2(n-1)}{n}|P_u A||P_v A|.$$

Thus (Bezout) and (3B) don't hold for some convex sets A in $\mathbb{R}^n$, for $n \geq 3$.

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$c_2 = 1$, $c_3 = 4/3$ *and let* $\varphi = \frac{1+\sqrt{5}}{2}$ *be the golden ratio. Then, for* $n \ge 3$

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The optimal constants in these inequalities were computed recently by Brazitikos-Giannopoulos-Liakopoulos '18, Giannopoulos-Koldobsky-Valettas '18 and Alonso-Gutiérrez-Artstein-Avidan-González-Merino-Jiménez-Villa '19.

Plan

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All preceding inequalities hold in $\mathbb{R}^3$.

Method: We prove the inequality in its last form, using projections. Applying a linear transform we only need to prove that for any zonoid A in $\mathbb{R}^3$ one has

$$|A||P_{e_1,e_2}A| \leq |P_{e_1}A||P_{e_2}A|.$$

Proof of the (3B) inequality for zonoids in $\mathbb{R}^3$

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For any zonoid A *in* $\mathbb{R}^3$ *one has* $|A||P_{e_1,e_2}A| \leq |P_{e_1}A||P_{e_2}A|$.

Proof of the (3B) inequality for zonoids in $\mathbb{R}^3$

Theorem (F.-Madiman-Meyer-Zvavitch 2022+)

For any zonoid A in $\mathbb{R}^3$ one has $|A||P_{e_1,e_2}A| \leq |P_{e_1}A||P_{e_2}A|$.

Method: By approximation, it is enough to prove it for A zonotope: $A = \sum_{i=1}^{M}[0, u_i]$, where $u_i = (x_i, y_i, z_i) \in \mathbb{R}^3$, for $1 \leq i \leq M$.

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We fix $y = (y_1, \ldots, y_M)$ and $z = (z_1, \ldots, z_M)$ in $\mathbb{R}^M$ and we consider the inequality as a comparison of two convex functions of $x = (x_1, \ldots, x_M) \in \mathbb{R}^M$, that are moreover affine by parts.

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So we only need to prove the inequality at infinity and at the critical points of the above right hand side function.

Proof of the (3B) inequality for zonoids in $\mathbb{R}^3$

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The limit case reduces to Bonnesen's inequality in dimension 2. Then we use some intricate induction.

Proof of the (3B) inequality for zonoids in $\mathbb{R}^3$

Theorem (F.-Madiman-Meyer-Zvavitch 2022+)

For any zonoid A in $\mathbb{R}^3$ one has $|A||P_{e_1,e_2}A| \leq |P_{e_1}A||P_{e_2}A|$.

Method: By approximation, it is enough to prove it for A zonotope: $A = \sum_{i=1}^{M}[0, u_i]$, where $u_i = (x_i, y_i, z_i) \in \mathbb{R}^3$, for $1 \leq i \leq M$. Then we want to prove:

$$\sum_{1\leq i<j<k\leq M}\left|\det\begin{pmatrix} x_i & x_j & x_k \\ y_i & y_j & y_k \\ z_i & z_j & z_k \end{pmatrix}\right| \sum_{i=1}^{M}|z_i| \leq \sum_{1\leq i<j\leq M}\left|\det\begin{pmatrix} y_i & y_j \\ z_i & z_j \end{pmatrix}\right| \sum_{1\leq i<j\leq M}\left|\det\begin{pmatrix} x_i & x_j \\ z_i & z_j \end{pmatrix}\right|$$

We fix $y = (y_1, \ldots, y_M)$ and $z = (z_1, \ldots, z_M)$ in $\mathbb{R}^M$ and we consider the inequality as a comparison of two convex functions of $x = (x_1, \ldots, x_M) \in \mathbb{R}^M$, that are moreover affine by parts.
So we only need to prove the inequality at infinity and at the critical points of the above right hand side function.
The limit case reduces to Bonnesen's inequality in dimension 2. Then we use some intricate induction.

Remark: for zonoids, see also the recent nice vector valued Maclaurin inequalities put forward by Brazitikos-McIntyre 2021 and Joós-Lángi 2022+.

Plan

Courtade's conjecture

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1) Statement of the conjecture. Let B, C be compact convex sets in $\mathbb{R}^n$. Is it true that

$$(|B||C|)^{1/n} + (|B_2^n||B_2^n + B + C|)^{1/n} \leq (|B_2^n + B||B_2^n + C|)^{1/n}? \quad (CC)$$

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4) B zonoid

Theorem (F.-Madiman-Zvavitch 2022+)

Let B be a zonoid and C be a compact convex set in $\mathbb{R}^n$. Then

$$|B_2^n||B_2^n + B + C| \leq |B_2^n + B||B_2^n + C|$$

Proof of Courtade's conjecture in $\mathbb{R}^2$

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Squaring both sides we want to prove

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Expanding with mixed volumes, simplifying and squaring again, we rewrite it as

$$|A||B||C||A+B+C| \le \left(2V(A,B)V(A,C) + V(A,B)|C| + |B|V(A,C) - |A|V(B,C)\right)^2 .$$

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Replacing B by rB and simplifying by r^2, we are reduced to proving: $\alpha r^2 + \beta r + \gamma \geq 0$. Then we use that $\alpha = |B|^2(V^2(A,C) - |A||C|) \geq 0$ and, after some painful calculation, the discriminant is

$$\Delta = c\left((|A|V(B,C) - V(A,B)V(A,C))^2 - \left(V(A,B)^2 - |A||B|\right)\left(V(A,C)^2 - |A||C|\right)\right) \leq 0,$$

where $c = |B|^2|A||C||A+C|$, and where the last inequality follows from Fenchel's inequality.

Open questions

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2) 3 zonoids' conjecture: Let $n \geq 4$. For any zonoids A, B, C in $\mathbb{R}^n$ do we have

$$|A + B + C||A| \leq |A + B||A + C|? \quad (3B)$$

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5) Is the golden ratio the optimal constant? Recall that the best constant c_n such that for any compact convex sets A, B, C in $\mathbb{R}^n$:

$$\mathrm{vol}_n(A + B + C)\mathrm{vol}_n(A) \leq c_n \mathrm{vol}_n(A + B)\mathrm{vol}_n(A + C),$$

satisfy $(4/3)^n \leq c_n \leq \varphi^n$.

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satisfy $(4/3)^n \leq c_n \leq \varphi^n$. Question: do we have $c_n^{1/n} \to \varphi$?

End

Thank you!