# Isoperimetric problems for zonotopes

Zsolt Lángi (joint work with Antal Joós)

HAS-BUTE Morphodynamics Research Group
Department of Geometry, Budapest University of Technology and Economics, Hungary

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# Motivation



## Definition

The Minkowski sum of finitely many segments in $\mathbb{R}^d$ is called a zonotope.

## Theorem (McMullen 1971, Shephard 1974)

*If $Z = \sum_{i=1}^{n} [o, p_i]$, where $1 \leq i \leq n$, is a zonotope in $\mathbb{R}^d$, then*

$$V_d(Z) = \sum_{1 \leq i_1 < i_2 < \ldots < i_d \leq n} |p_{i_1} \wedge p_{i_2} \wedge \ldots \wedge p_{i_d}|.$$

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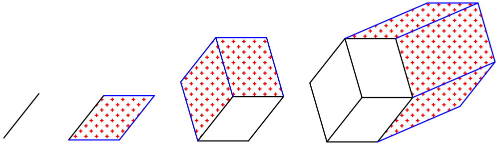

FIGURE: Decomposition of a zonotope $Z$ into parallelotopes generated by the generating segments of $Z$

**THEOREM (SHEPHARD 1974)**

*If $Z = \sum_{i=1}^{n} [o, p_i]$, where $1 \leq i \leq n$, then $Z$ can be decomposed into a family $\mathcal{F}$ of parallelotopes such that each element of $\mathcal{F}$ is a translate of a $d$-dimensional parallelotope $P = \sum_{j=1}^{d} [o, p_{i_j}]$ for some $1 \leq i_1 < i_2 < \ldots < i_d \leq n$, and for every such parallelotope $P$ there is a unique element of $\mathcal{F}$ which is a translate of $P$.*

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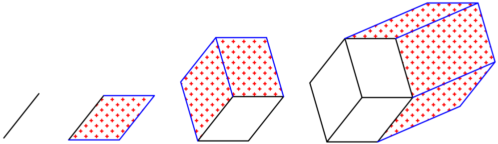

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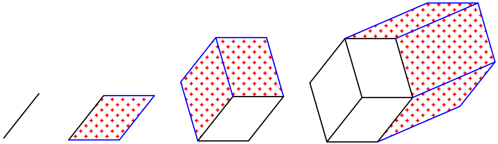

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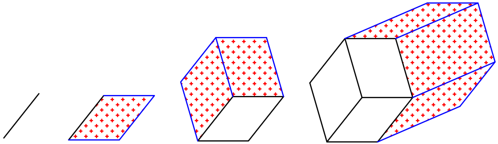

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# Notation



Let $d \geq 2$.

1. $Z = \sum_{i=1}^{n} [o, p_i]$ is a zonotope with $p_i \in \mathbb{R}^d$ for all values of $i$.
2. $\mathcal{P}^Z = \{p_1, \ldots, p_n\}$, for any $0 \leq k \leq d$, $\mathcal{P}_k^Z$: the family of $k$-element subsets of $\mathcal{P}^Z$ containing linearly independent vectors.
3. For any $I \in \mathcal{P}_k^Z$ with $k \geq 1$, $P(I) = \sum_{i \in I} [o, p_i]$; for $k = 0$, $P(\emptyset) = \{o\}$ (regarded as a 0-dimensional parallelotope)
4. for any $P(I)$ with $I \in \mathcal{P}_k^Z$, $B^{\perp}(I) := \mathbf{B}^d \cap (\operatorname{aff} P(I))^{\perp}$ and $\mathbb{S}^{\perp}(I) := \mathbb{S}^{d-1} \cap (\operatorname{aff} P(I))^{\perp}$, where $(\operatorname{aff} P(I))^{\perp}$ denotes the orthogonal complement of $\operatorname{aff} P(I)$.

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# Generalized decomposition theorem



**Theorem (Joós, L. 2022)**

*For any $t \geq 0$, the set $Z + t\mathbf{B}^d$ can be decomposed into a family $\mathcal{F}_Z$ of mutually non-overlapping convex bodies of the form $X + tB_X$ such that*

1. *for any $X + tB_X \in \mathcal{F}_Z$, $X$ is a translate of some parallelotope $P(I)$ with $I \in \mathcal{P}_k^Z$ for some $0 \leq k \leq d$, and $B_X \subseteq B^{\perp}(I)$ is the convex hull of $o$ and a spherically convex, compact subset of $S^{\perp}(I)$;*
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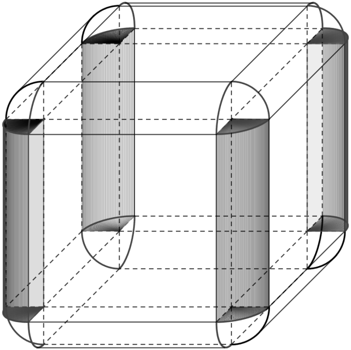

Figure: The body $Z + t\mathbf{B}^d$ if $Z$ is a cube generated by 3 mutually orthogonal segments. There are 4 translates of every generating segment appearing as edges of $Z$. The solid bodies in the picture correspond to the sets $X + tB_X$, where $X$ is a translate of a fixed generating segment.

# GENERALIZED DECOMPOSITION THEOREM

cubical zonotope: any $d$ generating vectors are linearly independent (equivalently, any face is an affine cube)

**REMARK**

If $Z$ is cubical, then the sets $X$ ($X \subset \mathrm{bd}(Z)$) in the theorem are the proper faces of $Z$, and $B_X$ is the set of the outer normal vectors of $X$ of length at most one.

**COROLLARY**

*For any zonotope $Z = \sum_{i=1}^{n} [o, p_i]$ in $\mathbb{R}^d$, and any $0 \leq k \leq d$, the kth intrinsic volume of $Z$ is*

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Corollary is proved by Brazitikos and McIntyre in 2021 using an integral geometric formula.

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Two parts:

1. Isoperimetric problems for zonotopes in $\mathbb{R}^d$ generated by $d$ or $d+1$ segments (parallelotopes and rhombic dodecahedra).
2. Isoperimetric problems for zonotopes in $\mathbb{R}^d$ generated by $n \gg d$ segments (asymptotic estimates).

Examined geometric quantities: intrinsic volumes, inradius (minimal width) denoted by $\mathrm{ir}(\cdot)$, circumradius (diameter) denoted by $\mathrm{cr}(\cdot)$.

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Notation:

$\mathcal{Z}_{\mathrm{p}} := \mathcal{Z}_{d,d}$
$\mathcal{Z}_{\mathrm{rd}} := \mathcal{Z}_{d,d+1}$
$Z_{\mathrm{p}}^{\mathrm{reg}} \in \mathcal{Z}_{d,d}$ is a cube
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**THEOREM (BEZDEK 2000)**

*Let $1 \leq k \leq d$ be arbitrary. Among rhombic dodecahedra in $\mathbb{R}^d$ of unit inradius, the ones with minimal kth intrinsic volumes are the regular ones.*

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# Zonotopes with a few generating vectors

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# Zonotopes with a few generating vectors, Part 1

**Theorem (Joós, L. 2022)**

*Let $1 \leq k \leq d-1$. Then, for any $Z_i \in \mathcal{Z}_i$ with $V_d(Z_i) = V_d(Z_i^{\mathrm{reg}})$, where $i \in \{\mathrm{p}, \mathrm{rd}\}$, we have*

$$V_k(Z_i) \geq V_k(Z_i^{\mathrm{reg}})$$

*with equality if and only if $Z_i$ is congruent to $Z_i^{\mathrm{reg}}$.*
*Furthermore, we $\mathrm{cr}(Z_i) \geq \mathrm{cr}(Z_i^{\mathrm{reg}})$.*

**Corollary**

*For any $Z_i \in \mathcal{Z}_i$ with $\mathrm{ir}(Z_i) = \mathrm{ir}(Z_i^{\mathrm{reg}})$, where $i \in \{\mathrm{p}, \mathrm{rd}\}$, we have*

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# Zonotopes with a few generating vectors, Part 2



**Theorem (Joós, L. 2022)**

*If $Z \in \mathcal{Z}_{\mathrm{p}}$ satisfies $V_1(Z) = V_1(Z_{\mathrm{p}}^{\mathrm{reg}})$, then $\mathrm{cr}(Z) \geq \mathrm{cr}(Z_{\mathrm{p}}^{\mathrm{reg}})$, with equality if and only if $Z$ is a cube.*

**Theorem (Joós, L. 2022)**

*Let $d \geq 2$. Then for any $Z = \sum_{i=1}^{d+1} [o, p_i] \in \mathcal{Z}_{\mathrm{rd}}$ satisfying $\sum_{i=1}^{d+1} p_i = o$ and $V_1(Z) = V_1(Z_{\mathrm{rd}}^{\mathrm{reg}})$, we have*

$$\mathrm{cr}(Z) \geq \mathrm{cr}(Z_{\mathrm{rd}}^{\mathrm{reg}}),$$

*with equality if and only if $Z$ is congruent to $Z_{\mathrm{rd}}^{\mathrm{reg}}$. Furthermore, if $d$ is odd, then there is a rhombic dodecahedron $Z' = \sum_{i=1}^{d+1} [o, p_i']$ with $V_1(Z') = V_1(Z_{\mathrm{rd}}^{\mathrm{reg}})$ and $\mathrm{cr}(Z') < \mathrm{cr}(Z_{\mathrm{rd}}^{\mathrm{reg}})$.*

# ZONOTOPES WITH A FEW GENERATING VECTORS, PART 2

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# Zonotopes with a few generating vectors, Part 3



**Proposition (Joós, L. 2022)**

*For any $Z \in \mathcal{Z}_\mathrm{p}$ with $V_1(Z) = V_1(Z_\mathrm{p}^\mathrm{reg})$, we have $V_2(Z) \leq V_2(Z_\mathrm{p}^\mathrm{reg})$, with equality if and only if $Z$ is a cube.*

**Theorem (Joós, L. 2022)**

*Let $Z_\mathrm{rd}^\mathrm{reg} = \sum_{i=1}^{d+1} [o, q_i]$, where $q_i \in \mathbb{S}^{d-1}$ for all values of $i$. Then, if $Z = \sum_{i=1}^{d+1} [o, p_i]$ is a rhombic dodecahedron with $p_i \in \mathbb{S}^{d-1}$ for all values of $i$, then*

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**Total squared $k$-content of a simplex**: sum of the squares of the $k$-volumes of all $k$-faces

### THEOREM (TANNER 1974)

*Let $2 \leq k \leq d$. Among simplices in $\mathbb{R}^d$ with a given total squared 1-content, the ones with maximal total squared $k$-content are the regular ones.*

$\sigma_m^k(x_1, x_2, \ldots, x_m)$: elementary symmetric polynomial of degree $k$ with the variables $x_1, x_2, \ldots, x_k$

### LEMMA (MACLAURIN'S INEQUALITY)

*Let $1 \leq k < m$ be integers, and $x_1, \ldots, x_m > 0$ be positive real numbers. Then*

$$\left( \frac{\sigma_m^k(x_1, x_2, \ldots, x_m)}{\binom{m}{k}} \right)^{\frac{1}{k}} \geq \left( \frac{\sigma_m^{k+1}(x_1, x_2, \ldots, x_m)}{\binom{m}{k+1}} \right)^{\frac{1}{k+1}}.$$

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# Zonotopes with a few generating vectors, Part 4



Total squared $k$-content of a simplex: sum of the squares of the $k$-volumes of all $k$-faces

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### CONJECTURE (BRAZITIKOS, MCINTYRE 2021)

*Let $x_1, x_2, \ldots, x_n \in \mathbb{R}^d$ be given with $1 \leq d \leq n$. Then for any $p \in [0, \infty]$ and $2 \leq k \leq d$, we have*

$$\left( \frac{\sum_{1 \leq i_1 < \ldots < i_k \leq n} |x_{i_1} \wedge x_{i_2} \wedge \ldots \wedge x_{i_k}|^p}{\binom{n}{k}} \right)^{\frac{1}{pk}} \leq$$

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*with equality if and only if $n = d$ and the vectors form an orthonormal basis.*

Proved for $p = 0$ and $p = \infty$, for $p = 2$ and $n = d$, and for $p = 1$, $n = d$ and $k = 2, 3, d$.

# Zonotopes with a few generating vectors, Part 4

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Proved for $p = 0$ and $p = \infty$, for $p = 2$ and $n = d$, and for $p = 1$, $n = d$ and $k = 2, 3, d$.

Let $Z = \sum_{i=1}^{n} [o, x_i] \subset \mathbb{R}^d$.

$$V_{k,p}(Z) = \sum_{1 \leq i_1 < \ldots < i_k \leq n} |x_{i_1} \wedge x_{i_2} \wedge \ldots \wedge x_{i_k}|^p$$

**Theorem**

*Let $Z$ be a rhombic dodecahedron in $\mathbb{R}^d$. Then, for any $1 \leq k < m \leq d$, the quantity*

$$\frac{\left(V_{k,2}(Z)\right)^m}{\left(V_{m,2}(Z)\right)^k}$$

*is minimal if and only if $Z$ is regular.*

# Zonotopes with a few generating vectors, Part 4

Let $Z = \sum_{i=1}^{n} [o, x_i] \subset \mathbb{R}^d$.

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**Theorem**

*Let $Z$ be a rhombic dodecahedron in $\mathbb{R}^d$. Then, for any $1 \le k < m \le d$, the quantity*

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**Problem (Betke, McMullen 1983)**

*For any $\varepsilon > 0$, find the smallest number $N = N(\varepsilon)$ such that the Euclidean ball can be approximated within error $\varepsilon$ (in Hausdorff distance) by a zonotope generated by $N$ segments.*

**Theorem (Bourgain, Lindenstrauss 1988 and 1993, Linhart 1989, Bourgain, Lindenstrauss, Milman 1989, Matoušek 1996)**

*There are constants $c, C > 0$ depending only on the dimension $d$ such that*

$$c\varepsilon^{\frac{-2(d-1)}{d+2}} \leq N(\varepsilon) \leq \begin{cases} C\varepsilon^{\frac{-2(d-1)}{d+2}}, & \text{if } d = 2 \text{ or } d \geq 5, \\ C\left(\varepsilon^{-2}\log|\varepsilon|\right)^{\frac{(d-1)}{d+2}}, & \text{otherwise.} \end{cases}$$

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*Let* $1 \leq i < k \leq d$. *Then there are positive constants* $c, C$ *depending only on* $d$ *such that for any sufficiently large value of* $n$,

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# APPLICATION I

$\ell_1$-polarization problem:

For a multiset $\omega_n = \{x_1, x_2, \ldots, x_n\}$ in $\mathbb{S}^{d-1}$, the $\ell_1$*-polarization of* $\omega_n$ is defined as

$$M_1(\omega_n) = \max\left\{\sum_{i=1}^{n} |\langle x_i, u\rangle| : u \in \mathbb{S}^{d-1}\right\},$$

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$$V_1(Z) = \sum_{i=1}^{n} |p_i|,$$

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*For any $n \geq d \geq 1$, find the minimal circumradius of all equilateral zonotopes in $\mathcal{Z}_{d,n}$ with a given mean width.*

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## Remark

It was shown by Ambrus and Nietert in 2019 that

$$M_n^1(\mathbb{S}^{d-1}) = n\mu_{d,1} + o\left(\frac{n}{\sqrt{d}}\right)$$

if $n, d \to \infty$ and $n = \omega\left(d^2 \log d\right)$, where $\mu_{d,1} = \frac{\Gamma\left(\frac{d}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{d+1}{2}\right)}$.

Our theorems yield that

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## CONJECTURE (BRAZITIKOS, MCINTYRE 2021)

*Let $x_1, x_2, \ldots, x_n \in \mathbb{R}^d$ be given with $1 \leq d \leq n$. Then for any $p \in [0, \infty]$ and $2 \leq k \leq d$, we have*

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*with equality if and only if $n = d$ and the vectors form an orthonormal basis.*

Proved for $p = 0$ and $p = \infty$, for $p = 2$ and $n = d$, and for $p = 1$, $n = d$ and $k = 2, 3, d$.

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*with equality if and only if $n = d$ and the vectors form an orthonormal basis.*

Proved for $p = 0$ and $p = \infty$, for $p = 2$ and $n = d$, and for $p = 1$, $n = d$ and $k = 2, 3, d$.

# Application II



## Conjecture (Brazitikos, McIntyre 2021)

*Let $x_1, x_2, \ldots, x_n \in \mathbb{R}^d$ be given with $1 \leq d \leq n$. Then for any $p \in [0, \infty]$ and $2 \leq k \leq d$, we have*

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# Application II



### Theorem

*Let $n \geq d$ and $Z = \sum_{i=1}^{n} [o, x_i]$ be a zonotope in $\mathbb{R}^d$. Then, for any $1 \leq k < d$, the quantity*

$$\left( \frac{V_{k,2}(Z)}{\binom{n}{k}} \right)^{\frac{1}{k}} \geq \left( \frac{V_{k+1,2}(Z)}{\binom{n}{k+1}} \right)^{\frac{1}{k+1}},$$

*with equality if and only if $n = d$ and $Z$ is a cube, or if the dimension of $Z$ is at most $k-1$.*

# Application II

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# Gracias por su atención