Löwner's problem for log-concave functions and beyond

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joint with

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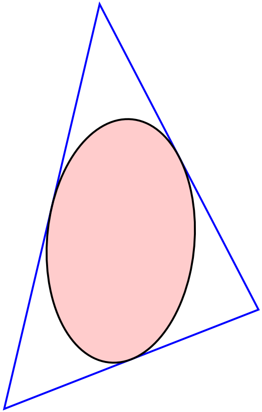
Convex sets

The John ellipsoid: K and $\mathbf{B}^d$

1. K is a convex body in $\mathbb{R}^d$
2. The *John ellipsoid* of K is the maximal volume ellipsoid contained in K



John's condition

$K \subset \mathbb{R}^d$ convex body.

F. John's theorem '48 (+ K. Ball'92)

Assume $\mathbf{B}^d \subseteq K$. TFAE:

1. $\mathbf{B}^d$ is the maximum volume ellipsoid contained in K
2. there are contact points $u_1, \ldots, u_m \in \operatorname{bd} \mathbf{B}^d \cap \operatorname{bd} K$ and weights $c_1, \ldots, c_m > 0$ such that

$$\sum_{i=1}^{m} c_i u_i = 0 \quad \text{and} \quad \sum_{i=1}^{m} c_i u_i \otimes u_i = I,$$

where I is the identity operator on $\mathbb{R}^d$.

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where I is the identity operator on $\mathbb{R}^d$.

Löwner ellipsoid: The minimum volume ellipsoid containing K.

Functions

A more general setting: log-concave functions

Definition

$f : \mathbb{R}^d \to \mathbb{R}_+$ is *log-concave* if for all $x, y \in \mathbb{R}^d$ and $\lambda \in [0, 1]$

$$f((1-\lambda)x + \lambda y) \geq f^{1-\lambda}(x) f^{\lambda}(y).$$

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Examples: $\{$cvx. bodies$\} \hookrightarrow \{$log-conc. functions$\}$

1. $\chi_K = \begin{cases} 1, & \text{if } x \in K \\ 0, & \text{otherwise} \end{cases}$.
2. Assume $0 \in \operatorname{int}(K)$, and $t \geq 1$. Take $f(x) = e^{-\|x\|_K^t}$.
3. Eg., *Gaussian distribution* on $\mathbb{R}^d$: $f(x) = (2\pi)^{-d/2} e^{-|x|^2/2}$.
4. The *marginal* on $\mathbb{R}^d$ of the uniform measure on a convex body in $\mathbb{R}^{d+s}$.

The functional John problem

How to phrase it?

Given $f : \mathbb{R}^d \longrightarrow \mathbb{R}_+$ log-concave function.

$$\max_{g \in ???} \int_{\mathbb{R}^d} g \quad \text{subject to} \quad g \leq f.$$

What function class should ??? be?

The AMJV ellipsoid – Definition

Alonso-Gutiérrez, Merino, Jiménez, Villa 2018:

$f : \mathbb{R}^d \to \mathbb{R}_+$ log-concave, integrable.

1. Super-level sets: for $\alpha \in (0, \|f\|]$, consider the convex body
$$\{x \in \mathbb{R}^d \ : \ f(x) \geq \alpha\}.$$
2. Take the John ellipsoid in it, $E(\alpha)$.
3. Consider $\alpha \cdot \chi_{E(\alpha)}$.

Clearly, $\alpha \cdot \chi_{E(\alpha)} \overset{pw}{\leq} f$.

The AMJV ellipsoid – Properties

Existence, uniqueness [AMJV'18]

There is a unique $\alpha \in (0, \|f\|]$ st.

$$\int_{\mathbb{R}^d} \alpha \cdot \chi_{E(\alpha)} \,\mathrm{d}x = \alpha \cdot \mathrm{vol}_d\, E(\alpha)$$

is maximal.
Moreover, $\alpha \geq e^{-d}\,\|f\|$.

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$^{(0)}J_f = \alpha \cdot \chi_{E(\alpha)}$ — the AMJV ellipsoid of f.

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Volume (integral) ratio

$$\left(\frac{\int_{\mathbb{R}^d} f \, \mathrm{d}x}{\int_{\mathbb{R}^d} {}^{(0)}J_f \, \mathrm{d}x} \right)^{1/d} \leq \Theta\sqrt{d}$$

And the extremum is characterized.

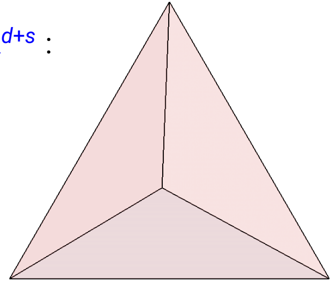
Motivation

We want a bit more

- $\mu_{\mathbf{B}^{d+s}}\big|_{\mathbb{R}^d}$ — Why not an ellipsoid?
- Limit case: Normal distribution — Why not an ellipsoid?
- Vertical cylinder under the graph — not an ellipsoid.
- John's characterization?

Geometric motivation

$\mathbb{R}^{d+s}$:



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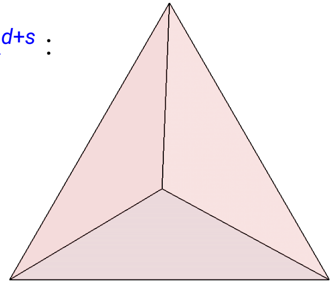

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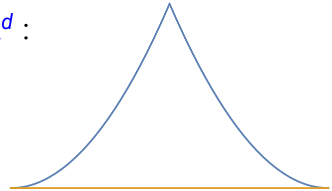

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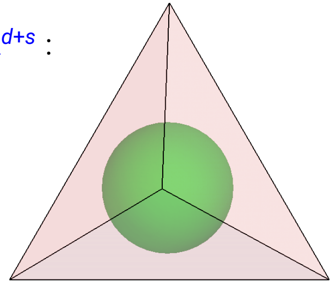

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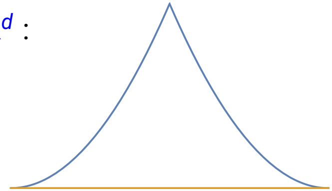

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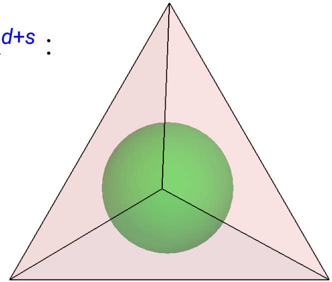

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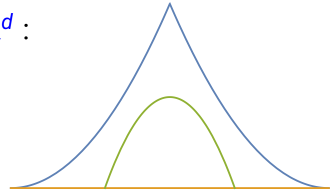

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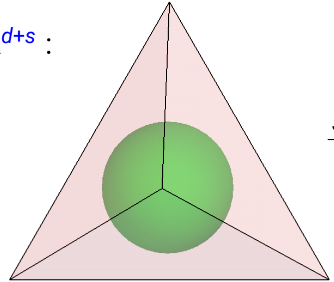
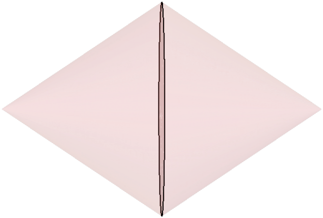


$\xrightarrow{\textit{Schwarz}}$



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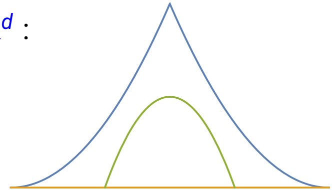

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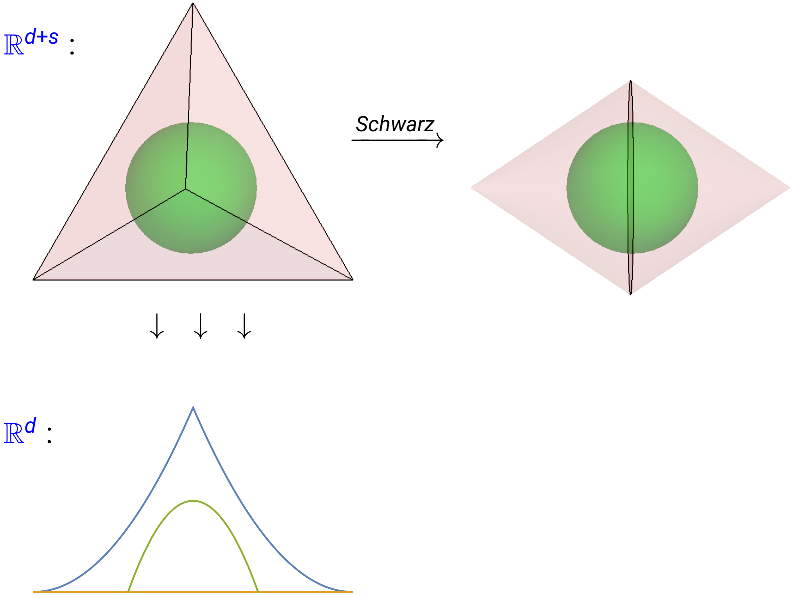

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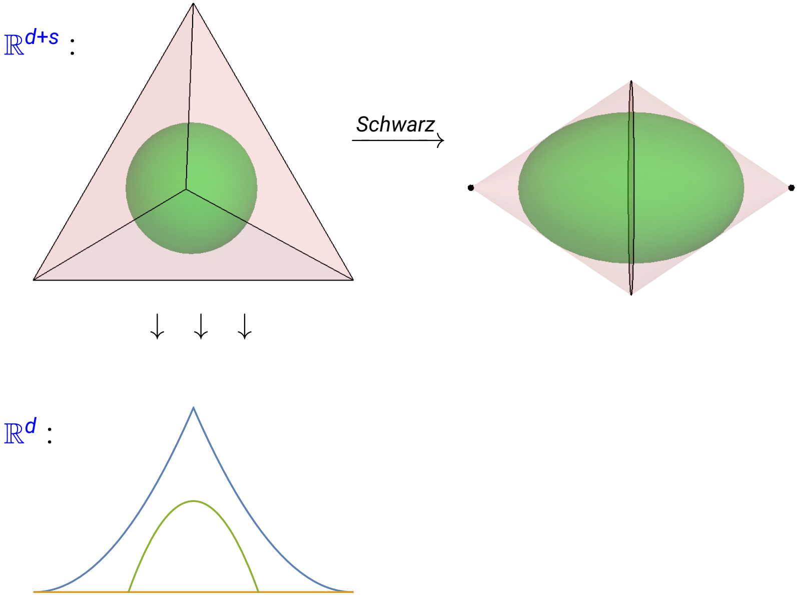

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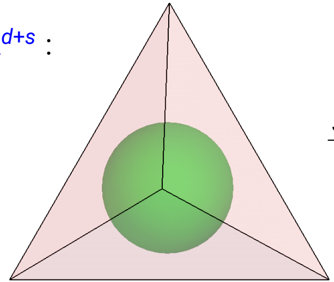

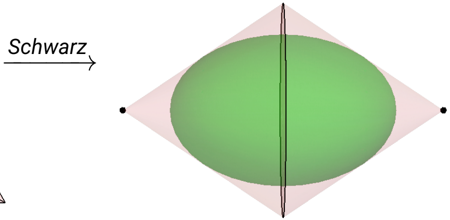

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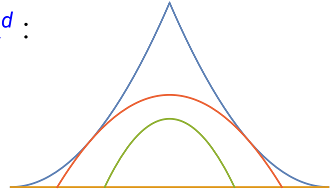

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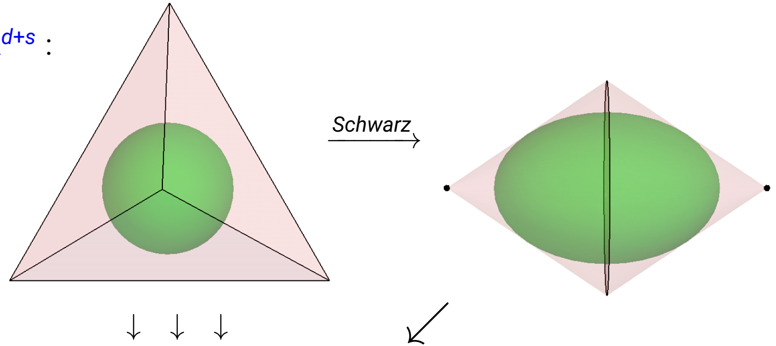

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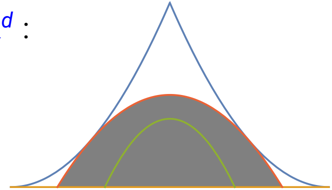

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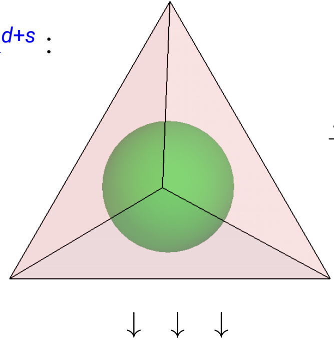

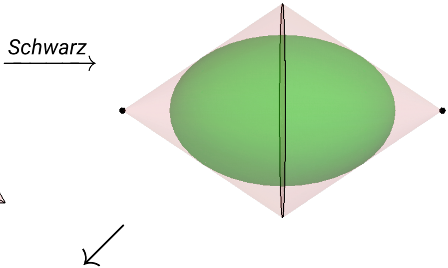

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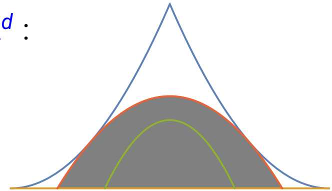

Problems:

1. s may be large ($\to \infty$).
2. $s \in \mathbb{Z}$.

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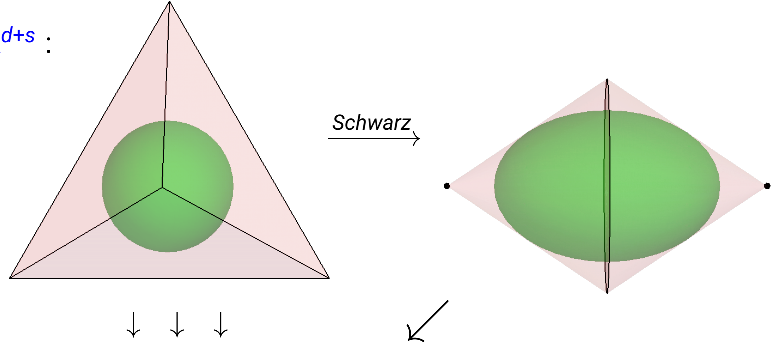

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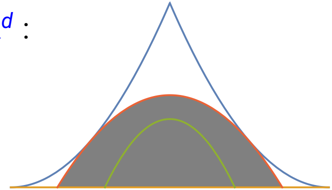

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Solution: Compress s dimensions into 1.

Our Definition of John's problem

Positions of the height function of $\mathbf{B}^{d+1}$

Fix $s > 0$.
Positions of a function h:

$$\mathcal{E}[h] = \left\{\alpha h(Ax + a) \ : \ \alpha \in \mathbb{R}_+, A \in GL(d), a \in \mathbb{R}^d\right\}.$$

Functional John s-problem

Given $f : \mathbb{R}^d \longrightarrow \mathbb{R}_+$ log-concave function.

$$\max_{g \in \mathcal{E}[h_{\mathbf{B}}]} \int_{\mathbb{R}^d} g^s \quad \text{subject to} \quad g \leq f,$$

where $h_{\mathbf{B}} : \mathbb{R}^d \longrightarrow \mathbb{R}_+$ is the height function of the ball $\mathbf{B}^{d+1}$:

$$h_{\mathbf{B}}(x) = \begin{cases} \sqrt{1 - x^2}, & \text{if } x^2 \leq 1, \\ 0, & \text{otherwise.} \end{cases}$$

Existence and uniqueness

Existence and uniqueness [G. Ivanov, MN]

Let $s > 0$ and f be a proper log-concave function on $\mathbb{R}^d$. Then, there exists a unique John s-ellipsoid of f.
Moreover,

$$\left\| {}^{(s)}J_f \right\| \geq e^{-d} \left\| f \right\| .$$

proper — upper-semicontinuous with finite positive integral.

John's condition

Theorem (G. Ivanov, MN)

Assume $h_{\mathbf{B}} \leq f$. TFAE:

1. $h_{\mathbf{B}}$ is the unique solution of the Functional John s-problem for f.
2. there are contact points $u_1, \ldots, u_m \in \mathbf{B}^d$ (ie., $h_{\mathbf{B}}(u_i) = f(u_i)$) and weights $c_1, \ldots, c_m > 0$ such that

$$\sum_{i=1}^{m} c_i u_i = 0 \quad \text{and} \quad \sum_{i=1}^{m} c_i f(u_i)^2 = s, \quad \text{and} \quad \sum_{i=1}^{m} c_i u_i \otimes u_i = I.$$

Further properties of John s-functions [G.Ivanov, MN]

- $s \longrightarrow 0$: We recover The AMJV ellipsoid.
- $s \longrightarrow \infty$: We obtain the Gaussian distribution (WARNING: If it exists, No uniqueness).

Theorem (Functional Bárány–Katchalski–Pach)

Let $f_1, \ldots, f_n$ be upper semi-continuous log-concave functions on $\mathbb{R}^d$. For every $\sigma \subseteq [n]$, let f_σ denote the pointwise minimum:

$$f_\sigma(x) = \min\{f_i(x) \ : \ i \in \sigma\}.$$

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Then there is a set $\sigma \in \binom{[n]}{\leq 3d+2}$ of at most $3d+2$ indices such that, with the notation $f = f_{[n]}$, we have

$$\int_{\mathbb{R}^d} f_\sigma \leq 100^d d^{2d} \int_{\mathbb{R}^d} f.$$

Polarity, Löwner problem

For convex sets $K \subset \mathbb{R}^d$ convex set. Then its *polar* is

$$K^\circ = \left\{ y \in \mathbb{R}^d \ : \ \langle x, y \rangle \leq 1 \text{ for all } x \in K \right\}.$$

For log-concave functions

$f : \mathbb{R}^d \to \mathbb{R}_+$ log-concave function. Then its *polar* (or *log-conjugate*) function is

$$f^\circ(y) = \inf_{x \in \mathrm{supp}(f)} \frac{e^{-\langle x, y \rangle}}{f(x)}.$$

Artstein-Avidan – Klartag – Milman (2004): Functional Santaló inequality.

G. Ivanov, I. Tsiutsiurupa (2021): Löwner s-problem with "outer" function $\ell = h^\circ$.

Convex sets again

K and L convex bodies in $\mathbb{R}^d$

Largest volume affine image of K in L?

Definition

K is in *John's position* in L if $K \subseteq L$ and

$$\sum_{i=1}^{m} c_i u_i \otimes v_i = I_d. \tag{1}$$

and

$$\sum_{i=1}^{m} c_i u_i = \sum_{i=1}^{m} c_i v_i = 0 \tag{2}$$

for some $u_i \in \operatorname{bd} L \cap \operatorname{bd} K$, $v_i \in \operatorname{bd} L^\circ \cap \operatorname{bd} K^\circ$ and $c_i \in \mathbb{R}^+$, where $\langle u_i, v_i \rangle = 1$.

Generalization of John's Condition

Lewis (1979), Milman (1989), Giannopoulos – Perissinaki – Tsolomitis (2001), Bastero – Romance (2002), Gordon – Litvak – Meyer – Pajor (2004).

Theorem (GLMP, 2004)

If K is in a position of maximal volume in L, then there exist $z \in \operatorname{int}(K)$ such that $K - z$ is in John's position in $L - z$ with $m \leq d^2 + d$.

Containment (GLMP, 2004)

If $K \subseteq L$ is in John's position, then $L \subseteq -dK$.

Back to functions

John condition for f and g

John s-problem:

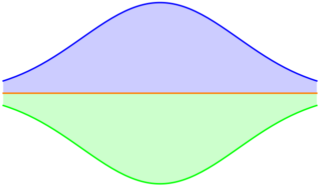
$f, g \colon \mathbb{R}^d \to \mathbb{R}_+$ proper log-concave functions,
g of compact support.

$$\max_{h \in \mathcal{E}[g]} \int_{\mathbb{R}^d} h^s \quad \text{subject to} \quad h \le f, \qquad (3)$$

where $\mathcal{E}[g] = \left\{ \alpha g(Ax + a) \ : \ \alpha \in \mathbb{R}_+, A \in GL(d), a \in \mathbb{R}^d \right\}$.

Lifting:

$$\mathrm{Lift}\,(f) = \{(x, \xi) \in \mathbb{R}^{d+1} \colon x \in \mathrm{cl}\,(\mathrm{supp}\,(f))\,, |\xi| \le f(x)\}.$$



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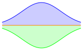

Contacts: $(u, \bar{v})$, where

$$u \in \mathrm{cl}\,(\mathrm{supp}\,(g))\,, \quad g(u) = f(u),$$

$$\bar{v} \in N(\mathrm{Lift}\,(f), (u, f(u))), \quad \langle \bar{v}, (u, f(u)) \rangle = 1.$$

John condition for f and g

Theorem (G. Ivanov, I. Tsiutsiurupa, MN)

Assume that $g \leq f$, and some technical conditions. Then

1. If g is a local maximizer, then there are contacts $(u_1, \overline{v}_1), \ldots, (u_m, \overline{v}_m)$ such that

$$\sum_{i=1}^{m} c_i u_i \otimes v_i = I_d, \quad \sum_{i=1}^{m} c_i f(u_i) \cdot |\overline{v}_i - v_i| = s \quad \text{and} \quad \sum_{i=1}^{m} c_i v_i = 0.$$

2. If there exist contacts satisfying the equations, then g is a global maximizer in Positive position John s-problem.

Positive position:
$\mathcal{E}^+[g] = \left\{ \alpha g(Ax + a) \ : \ \alpha \in \mathbb{R}_+, A \in GL(d) \text{ pos. definite}, a \in \mathbb{R}^d \right\}$.

Löwner conditions for f and g

Löwner s-problem:

$f, g\colon \mathbb{R}^d \to \mathbb{R}_+$ proper log-concave functions.

$$\min_{h \in \mathcal{E}[g]} \int_{\mathbb{R}^d} h^s \quad \text{subject to} \quad f \leq h,$$

where $\mathcal{E}[g] = \left\{ \alpha g(Ax + a) \ : \ \alpha \in \mathbb{R}_+, A \in GL(d), a \in \mathbb{R}^d \right\}$.

Theorem (G. Ivanov, I. Tsiutsiurupa, MN)

Assume that g° is compact support (ie., g is of log-linear growth), and $f \leq g$. Then, the following hold:

1. If g a local minimizer, then there are contacts of g° and f°:

$$\sum_{i=1}^{m} c_i u_i \otimes v_i = I_d, \quad \sum_{i=1}^{m} c_i f(u_i) \cdot |\bar{v}_i - v_i| = s \quad \text{and}$$

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Thank you!