General measure extensions of projection bodies

With Applications to Zhang's Inequality

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June 23, 2022

Based on joint work with
Michael Roysdon & Artem Zvavitch

Introduction: Convex bodies & Volume.

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$$h_K(x) = \max_{y \in K} \langle x, y \rangle.$$

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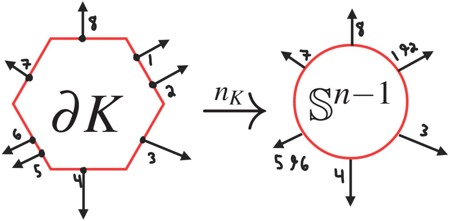

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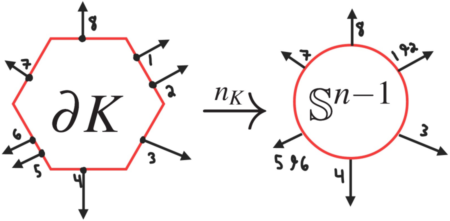

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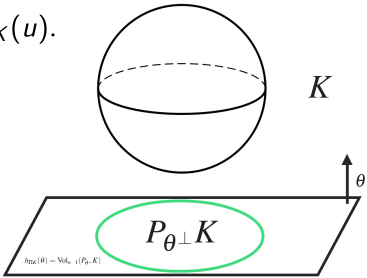
The *surface area measure* of K : for $E \subset \mathbb{S}^{n-1}$ Borel, $S_K(E) = \int_{n_K^{-1}(E)} dy$.

Cauchy's Integral Formula

If $P_{\theta^\perp}K$ denotes the orthogonal projection of K onto the hyperplane through the origin orthogonal to $\theta \in \mathbb{S}^{n-1}$, then *Cauchy's integral formula* states

$$\mathrm{Vol}_{n-1}\left(P_{\theta^\perp}K\right) = \frac{1}{2}\int_{\mathbb{S}^{n-1}} |\langle \theta, u\rangle| dS_K(u).$$



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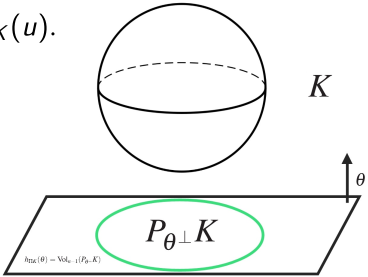

The above integral defines a norm (in θ). The **projection body** of K, denoted ΠK, is the symmetric convex body whose support function is given by $h_{\Pi K}(\theta) = \mathrm{Vol}_{n-1}\left(P_{\theta^\perp} K\right)$.

Affine Invariant Quantity

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$$\frac{1}{n^n}\binom{2n}{n} \quad \leq \quad \mathrm{Vol}_n(K)^{n-1}\mathrm{Vol}_n(\Pi^\circ K) \quad \leq \quad \left(\frac{\kappa_n}{\kappa_{n-1}}\right)^n.$$

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Equality occurs in Petty's inequality if, and only if, K is an ellipsoid.

Equality holds in Zhang's if, and only if, K is a simplex (convex hull of $n+1$ affinely independent points).

Covariogram

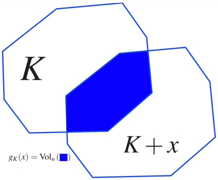
The **covariogram** of a convex body K is given by

$$g_K(x) = \mathrm{Vol}_n\left(K \cap (K+x)\right).$$

The support of $g_K(x)$ is the difference body of K, given by

$$DK = \{x : K \cap (K+x) \neq \emptyset\} = K + (-K),$$

The function $g_K^{1/n}$(and thus $\log(g_K)$) is concave on DK.



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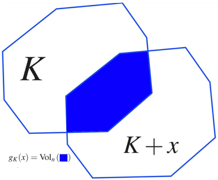

$$\left.\frac{\mathrm{d}g_K(r\theta)}{\mathrm{d}r}\right|_{r=0} = -h_{\Pi K}(\theta) = -\rho^{-1}_{\Pi^\circ K}(\theta)$$

Rogers-Shephard Inequality

$$2^n \leq \mathrm{Vol}_n(K)^{-1}\mathrm{Vol}_n(DK) \leq \binom{2n}{n}$$

The l.h.s. follows from the Brunn-Minkowski inequality, which asserts that the function $K \to \mathrm{Vol}_n(K)^{1/n}$ is concave on the class of convex bodies.

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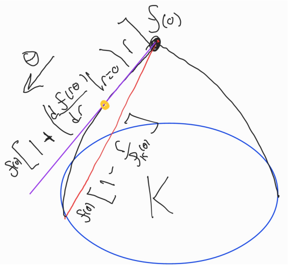
There is equality in the r.h.s. if, and only if, K is a simplex.

Facts of the Covariogram

For $\theta \in \mathbb{S}^{n-1}$, $r \in [0, \rho_{DK}(\theta)]$

$$0 \leq \left[1 - \frac{r}{\rho_{DK}(\theta)}\right] \leq \left(\frac{g_K(r\theta)}{\mathrm{Vol}_n(K)}\right)^{1/n} \leq \left[1 - \frac{r}{n\mathrm{Vol}_n(K)\rho_{\Pi^\circ K}(\theta)}\right]$$



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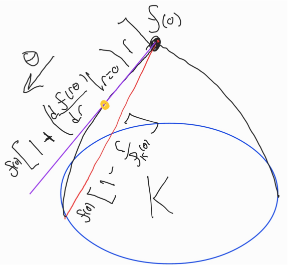

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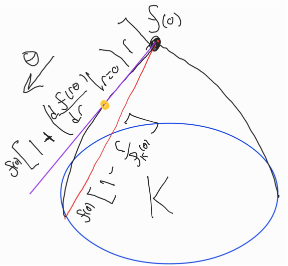
$\rho_{DK}(\theta) \leq n\text{Vol}_n(K)\rho_{\Pi^\circ K}(\theta) \iff DK \subseteq n\text{Vol}_n(K)\Pi^\circ K$. Equality occurs if, and only if $g_K^{1/n}$ is affine. This implies equality in the Brunn-Minkowski equality, implying $K \cap (K + x)$ and K are homothetic for $x \in DK$, which is a characterization of the simplex.



Classical proofs of the inequalities of Zhang and Rogers-Shephard

Use the translation invariance the Lebesgue measure

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Then, use a variable substitution, the radial function formula for volume and the Beta function $B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt$.

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4. Each generalization comes with a Berwald-type theorem.

Measure Properties

A measure μ is s-concave, $s > 0$, on a class $\mathcal{C}$ if every pair $A, B \in \mathcal{C}$ with positive measure and every $t \in [0, 1]$

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A measure μ is *F-concave* on a class $\mathcal{C}$ if there exists a continuous, invertible function F such that for

$$\mu((1-t)A + tB) \geq F^{-1}\left((1-t)F(\mu(A)) + tF(\mu(B))\right).$$

First generalization: Translated Averages

We first define the **translated-average** of K with respect to ν as:

$$\nu_\lambda(K) = \frac{1}{\mathrm{Vol}_n(K)} \int_K \nu(y - K)dy = \frac{1}{\mathrm{Vol}_n(K)} \int_{DK} g_K(x)d\nu(x).$$

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Notice we immediately obtain a weak Zhang's inequality

$$\nu_\lambda(K) = \frac{1}{\mathrm{Vol}_n(K)} \int_{DK} g_K(x)d\nu(x) \leq \nu(DK) \leq \nu(n\mathrm{Vol}_n(K)\Pi^\circ K).$$

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If one replaces ν with the Lebesgue measure, we see that we are missing the factor of $\binom{2n}{n}$. However, this result is asymptotically sharp by picking certain measures, e.g. $\nu = \gamma_n$.

Translated Averages: why just one?

The ν-translation of a convex body K averaged with respect to μ is given by

$$\nu_\mu(K) = \frac{1}{\mu(K)} \int_K \nu(y-K)d\mu(y) = \frac{1}{\mu(K)} \int_{DK} g_{\mu,K}(x)d\nu(x) \quad ,$$

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$g_{\mu,K}$ inherits* any concavity property of μ, e.g. $F\circ\mu$ concave implies $F\circ g_{\mu,K}$ concave.

Two Inequalities of Berwald type (L., Roysdon, Zvavitch)

f be a non-negative, concave function supported on L a convex body, and let h be an increasing, non-negative function. ν a Borel measure with density φ. If φ radially non-increasing, then

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If ϕ is radially non-decreasing and $\max_{x\in L} f(x) = f(0)$, then

$$\int_L h(f(x))\varphi(x)dx \leq \nu(\widetilde{L})n\int_0^1 h(f(0)t)(1-t)^{n-1}dt,$$

where $\widetilde{L}$ is the star body defined by $\rho_{\widetilde{L}}(\theta) = -\left(\left.\frac{\mathrm{d}f(r\theta)}{\mathrm{d}r}\right|_{r=0}\right)^{-1} f(0)$.

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There is equality in either case if, and only if, for every $\theta \in \mathbb{S}^{n-1}$, $\varphi(r\theta)$ is a constant and $f(r\theta) = \|f\|_\infty\left(1-\frac{r}{\rho_L(\theta)}\right)$. In this instance, $L = \widetilde{L}$.

Generalization of Rogers-Shephard's: the inequality

Theorem (L., Roysdon; 2022)

Let ν be a Borel measure with radially non-increasing density, and suppose μ is F-concave, $F:\mathbb{R}\to\mathbb{R}^+$ an increasing, invertible and differentiable function. Let K be a convex body. Then, one has

$$\frac{\nu_\mu(K)\mu(K)}{n\int_0^1 F^{-1}(F(\mu(K))t)(1-t)^{n-1}dt} \geq \nu(DK).$$

Additionally, if F is multiplicative, i.e. $F(ab)=F(a)F(b)$, then this becomes

$$\min\{\nu_\mu(K),\nu_\mu(-K)\}\left(n\int_0^1 F^{-1}(t)(1-t)^{n-1}dt\right)^{-1} \geq \nu(DK).$$

In particular, if $F(x)=x^s, s>0$, one obtains

$$\min\{\nu_\mu(K),\nu_\mu(-K)\}\binom{n+s^{-1}}{n} \geq \nu(DK).$$

Generalization Rogers-Shephard's: equality conditions

Theorem (L., Roysdon; 2022)

Equality occurs if, and only if, the following are true:

1. *If φ is the density of ν, then, for each $\theta \in \mathbb{S}^{n-1}$, $\varphi(r\theta)$ is independent of r and*
2. *for each $\theta \in \mathbb{S}^{n-1}$, $F \circ g_{\mu,K}(r\theta)$ is an affine function in the variable r.*

• We have shown the last equality condition implies K is a simplex if $F(x) = x^s, s > 0$.

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• They gave an example showing that ν having radially non-increasing density is necessary.

Generalizing the Projection Body

Let $f \in L^1(\mathbb{S}^{n-1})$. For μ Borel measure with density ϕ and a convex body K, $S_{\mu,K}$ is the Borel measure on $\mathbb{S}^{n-1}$ that satisfies

$$\int_{\partial K} f(n_K(y))\phi(y)dy = \int_{\mathbb{S}^{n-1}} f(u)dS_{\mu,K}(u).$$

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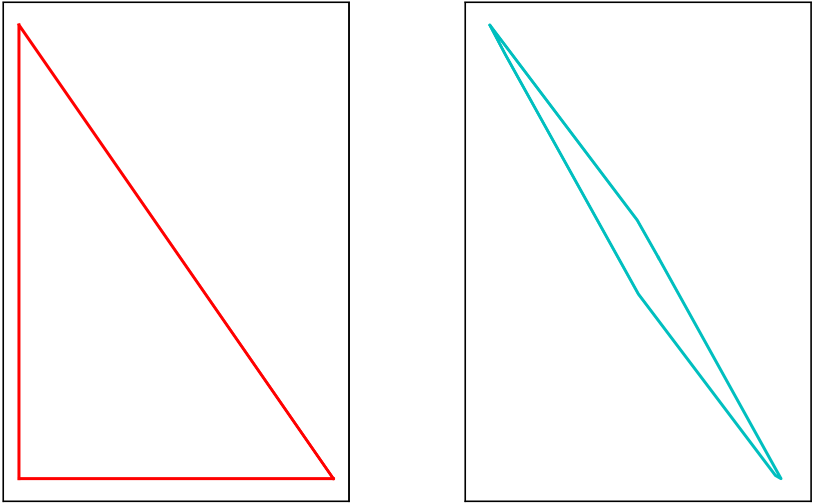
The projection body of K with respect to μ is the convex body $\Pi_\mu K$ whose support function is

$$h_{\Pi_\mu K}(\theta) = \frac{1}{2}\int_{\mathbb{S}^{n-1}} |\langle \theta, u\rangle| \, dS_{\mu,K}(u).$$



Left: A triangle/simplex $K \subset \mathbb{R}^2$. Right: $\Pi^{\circ}_{\gamma_2} K$.

The derivative

$q : \Omega \to \mathbb{R}$ is *Lipschitz* on a bounded domain Ω if, for every $x, y \in \Omega$, one has $|q(x) - q(y)| \leq C|x - y|$ for some $C > 0$.

Theorem (μ-brightness (L., Roysdon, Zvavitch; 2022))

Suppose the density of μ is locally Lipschitz. Then,

$$\left.\frac{\mathrm{d}g_{\mu,K}(r\theta)}{\mathrm{d}r}\right|_{r=0} = -\left(h_{\Pi_\mu K}(\theta) - \frac{1}{2}\int_K \langle \nabla\phi(y), \theta\rangle dy\right) = -h_{\Pi_\mu K - \eta_{\mu,K}}(\theta),$$

where we have defined the shift of K with respect to μ to be the vector $\eta_{\mu,K} = \frac{1}{2}\int_K \nabla\phi(y)dy$.

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where we have defined the shift of K with respect to μ to be the vector $\eta_{\mu,K} = \frac{1}{2}\int_K \nabla\phi(y)dy$.

Consider the case where μ is F-concave. Suppose $\eta_{\mu,K}$ is the origin (for example if μ has even density and K is symmetric).

$$DK \subseteq \frac{F(\mu(K))}{F'(\mu(K))}\Pi_\mu^\circ K.$$

Theorem (Zhang's inequality for positive-concavity (L., Roysdon, Zvavitch; 2022))

Suppose ν is a Borel measure with radially non-decreasing density and μ is Borel measure with Lipschitz density that is F-concave, F a non-negative, increasing, invertible function. Then, for K convex body

$$\nu_\mu(K) \leq \frac{n}{\mu(K)} \nu \left(\frac{F(\mu(K))}{F'(\mu(K))} \left(\Pi_\mu K - \eta_{\mu,K}\right)^\circ \right) \int_0^1 F^{-1}\left(F(\mu(K))t\right)(1-t)^{n-1} dt.$$

Equality occurs if, and only if, the following are true:

1. *If φ is the density of ν, then, for each $\theta \in \mathbb{S}^{n-1}$, $\varphi(r\theta)$ is independent of r,*
2. *for each $\theta \in \mathbb{S}^{n-1}$, $F \circ g_{\mu,K}(r\theta)$ is an affine function in the variable r, and*
3. *K is so that $\eta_{\mu,K} = 0$ and*

$$DK = \frac{F(\mu(K))}{F'(\mu(K))} \Pi_\mu^\circ K.$$

Special Cases

If μ is s-concave:

$$\binom{n+s^{-1}}{n}\nu_\mu(K) \leq \nu\left(s^{-1}\mu(K)\left(\Pi_\mu K - \eta_{\mu,K}\right)^\circ\right).$$

In this instance, the last two equality conditions are equivalent, and imply that K is a simplex whose shift is zero.

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If $\nu = \lambda$, the Lebesgue measure, then we have

$$\left(\int_0^1 F^{-1}[F(\mu(K))t](1-t)^{n-1}dt\right)^{-1} \le \left(\frac{F(\mu(K))}{F'(\mu(K))}\right)^n \frac{n\mathrm{Vol}_n\left(\left(\Pi_\mu K - \eta_{\mu,K}\right)^\circ\right)}{\mathrm{Vol}_n(K)\mu(K)}$$

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$\nu=\lambda$ and μ is $s\in(0,1/n]$, then

$$s^n\binom{n+s^{-1}}{n} \leq \frac{\mu^n(K)\mathrm{Vol}_n\left(\left(\Pi_\mu K-\eta_{\mu,K}\right)^\circ\right)}{\mathrm{Vol}_n(K)}.$$

Gaussian Measure

Theorem (The log-concave theorem (L., Roysdon, Zvavitch))

Let μ have locally Lipschitz density, $\mu(K) > 0$. Let $Q : (0, \infty) \to \mathbb{R}$ be an invertible, increasing function such that $\lim_{r\to 0+} Q(r) \in [-\infty, \infty)$ *and* $Q \circ g_{\mu,K}$ *is concave. Then, if* $Q'(\mu(K)) \neq 0$:

$$Vol_n(K) \leq \frac{n Vol_n\left((\Pi_\mu K - \eta_{\mu,K})^\circ\right)}{\mu(K)(Q'(\mu(K)))^n} \int_0^\infty Q^{-1}\left(Q(\mu(K)) - t\right) t^{n-1} dt.$$

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Corollary:

$$\boxed{\frac{1}{n!} \leq \frac{\gamma_n^n(K)\text{Vol}_n\left((\Pi_{\gamma_n} K - \eta_{\gamma_n,K})^\circ\right)}{\text{Vol}_n(K)}.}$$

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The Ehrhard inequality states that γ_n is Φ^{-1} concave on Borel sets, where $\Phi(x) = \gamma_1((-\infty, x))$.

Better Concavities for the Gaussian Measure

$Q(x) = \Phi^{-1}(x)$ satisfies the hypotheses of the log-concave theorem. Set $x = \Phi^{-1}(\gamma_n(K))$

$$\frac{1}{n!} \leq \frac{e^{nx^2/2}(2\pi\Phi(x)^2)^{(n+1)/2}}{\int_0^\infty z^n e^{-(z-x)^2/2} dz} \leq \frac{\gamma_n^n(K)\text{Vol}_n\left((\Pi_{\gamma_n}K - \eta_{\gamma_n,K})^\circ\right)}{\text{Vol}_n(K)}.$$

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The Gardner-Zvavitch inequality: for symmetric K, L and $t \in [0,1]$:

$$\gamma_n\left((1-t)K + tL\right)^{1/n} \ge (1-t)\gamma_n(K)^{1/n} + t\gamma_n(L)^{1/n}.$$

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- Conjectured by Gardner and Zvavitch in ('10)
- Counter examples constructed for non-symmetric bodies by Nayar and Tkocz ('12)
- Resolved in the affirmative for symmetric convex bodies by Eskenazis and Moschidis ('21).

Polarized Covariogram

If K is symmetric, then one sees that $K \cap (K + x)$ is not symmetric, but $(K - x/2) \cap (K + x/2)$ is symmetric.

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Can obtain the two Zhang-type inequalities shown (with $r_{\mu,K}$ in place of $g_{\mu,K}$).

E.g. the following *Polarized Zhang's Inequality for the Gaussian Measure* for a radially non-decreasing measure ν :

$$\binom{2n}{n} \nu_{\gamma_n}(K) \leq \nu\left(n\gamma_n(K)\Pi^{\circ}_{\gamma_n} K\right).$$