Directional Curvatures of Convex Bodies in $\mathbb{R}^n$

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geOmetry, anaLysis & convExity
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Motivation

How can we compute the curvature of a nonempty closed bounded convex set at a point on its boundary?

Introduction

In [GP] (V. Goncharov & F. Pereira, Neighbourhood Retractions of Nonconvex Sets in a Hilbert Space via Sublinear Functionals, J. Convex Anal., 2011) we proposed some concepts related to the (local) geometric structure of a closed bounded convex set F with zero in its interior, in a Hilbert space H.

The main numerical characteristic resulting from these considerations was the curvature of F.

Before showing our definition, we need some notations.

Introduction

Let us consider a Hilbert space H with the inner product $\langle \cdot, \cdot \rangle$ and the norm $\|\cdot\|$. For a closed bounded convex set $F \subset H$ with zero in its interior, we denote:

- F^o the polar set of F

$$F^o = \{\xi^* \in H : \langle \xi, \xi^* \rangle \leq 1 \ \ \forall \xi \in F\} ;$$

- $\mathfrak{J}_F(\cdot)$ the duality mapping $\mathfrak{J}_F : \partial F^o \to \partial F$

$$\mathfrak{J}_F(\xi^*) = \{\xi \in \partial F : \langle \xi, \xi^* \rangle = 1\} ,$$

where ∂F means the boundary of F;

- $\rho_F(\cdot)$ the Minkowski functional of F

$$\rho_F(\xi) = \inf \{\lambda > 0 : \xi \in \lambda F\} .$$

Definitions ([GP])

Let $\xi \in \partial F$ and $\xi^* \in \partial F^o$ such that $\langle \xi, \xi^* \rangle = 1$.

- The modulus of strict convexity of F at ξ w.r.t ξ^* is given by:

$$\widehat{\mathfrak{C}}_F(r, \xi, \xi^*) = \inf \{ \langle \xi - \eta, \xi^* \rangle : \eta \in \partial F, \|\xi - \eta\| \geq r \}, \quad r > 0.$$

- The set F is said to be strictly convex at ξ w.r.t. ξ^* if $\widehat{\mathfrak{C}}_F(r, \xi, \xi^*) > 0$ for all $r > 0$.

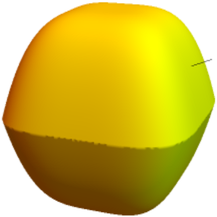
- The number

$$\hat{\varkappa}_F(\xi, \xi^*) = \liminf_{\substack{(r,\eta,\eta^*) \to (0+,\xi,\xi^*) \\ \eta \in \mathfrak{J}_F(\eta^*),\, \eta^* \in \partial F^o}} \frac{2\, \widehat{\mathfrak{C}}_F(r, \eta, \eta^*)}{\|\xi^*\|\, r^2}$$

is said to be the curvature of F at ξ w.r.t. ξ^*.

Introduction

So the curvature of F shows how rotund F is near a fixed boundary point ξ looking along a normal direction ξ^* (notice that $\xi^* \in \mathfrak{J}_F^{-1}(\xi) = \mathbf{N}_F(\xi) \cap \partial F^o$).



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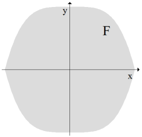
  - a characterization of the curvature of F in terms of the second derivative of its dual Minkowski functional $\rho_{F^o}(\cdot)$;
  - the well-posedness and regularity for some time-minimum problem.
- But, from a practical point of view, it is very difficult to use, even in $\mathbb{R}^2$, as we can see in the next example.

Introduction

An example

Consider the nonempty compact convex set

$$F := \left\{(x_1, x_2) \in \mathbb{R}^2 : |x_2| \leq 1 - x_1^4,\ -1 \leq x_1 \leq 1\right\}.$$



Fix $\xi := (\xi_1, \xi_2) \in \partial F$, with $\xi_1, \xi_2 > 0$, and (the unique) $\xi^* \in F^o$ such that $\langle \xi, \xi^* \rangle = 1$, which is given by

$$\xi^* = \frac{1}{1 + 3\xi_1^4} \left(4\xi_1^3, 1\right).$$

Even after a hard work we couldn't determine the exact value of the curvature, we only got

$$\frac{12\xi_1^2}{\sqrt{1 + 16\xi_1^6 \Sigma\left(\xi_1\right)}} \leq \hat{\varkappa}_F\left(\xi, \xi^*\right) \leq \frac{12\xi_1^2}{\sqrt{1 + 16\xi_1^6}},$$

where $\Sigma\left(\xi_1\right) := 1 + \left(\sum_{k=0}^{3} \left|\xi_1\right|^k\right)^2$.

So we started thinking about how we could get an easy formula to calculate the curvature. But Vladimir Goncharov died and the work stopped. After some time I came back to this idea, and I will present here what I got. So far I have only worked on $\mathbb{R}^n$, $n \geq 2$. (This work is dedicated to Vladimir's memory.)

Directional curvature

In everything that follows we consider a compact convex set $F \subset \mathbb{R}^n$, $n \geq 2$, with $\mathbf{0} \in \text{int}F$. We fix $\xi \in \partial F$ for which there are $\delta > 0$ and $f : \mathbb{R}^n \to \mathbb{R}$ of class $\mathcal{C}^2$ at $\xi + \delta \mathbf{B}$ ($\mathbf{B} \subset \mathbb{R}^n$ represents the open unit ball), such that

$$F \subset \{x \in \mathbb{R}^n : f(x) \leq 0\}, \qquad \langle \xi, \nabla f(\xi) \rangle > 0,$$

and such that for $x \in \xi + \delta \mathbf{B}$ we have $x \in \partial F$ if and only if $f(x) = 0$.

- Then there is $0 < \delta' \leq \delta$, such that for any $\eta \in \partial F \cap (\xi + \delta' \mathbf{B})$ we have

$$\nabla f(\eta) \neq \mathbf{0} \quad \text{and} \quad \mathbf{N}_F(\eta) = \bigcup_{\lambda \geq 0} \lambda \nabla f(\eta).$$

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- Consequently:
    - $\nabla f(\eta) \neq \mathbf{0}$ implies that there is a first $i \in I := \{1, ..., n\}$ such that

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$$f_{x_i}(\eta) := \frac{\partial f}{\partial x_i}(\eta) \neq 0,$$

  - and the unique $\eta^* \in \mathbf{N}_F(\eta)$ such that $\langle \eta, \eta^* \rangle = 1$ is given by

$$\eta^* := \frac{1}{\langle \eta, \nabla f(\eta) \rangle} \nabla f(\eta).$$

Directional curvature

Fixed such i, the tangent hyperplane to F at η is given by

$$\begin{aligned} \mathbf{T}_F(\eta) &= \{v \in \mathbb{R}^n : \langle v, \nabla f(\eta) \rangle = 0\} \\ &= \left\{ (v_1, \ldots, v_n) \in \mathbb{R}^n : v_i = -\sum_{j=1, j \neq i}^{n} \frac{f_{x_j}(\eta)}{f_{x_i}(\eta)} v_j \right\}. \end{aligned}$$

For any $j \in I \backslash \{i\}$ we define (the principal directions)

$$u^j(\eta) := \Bigg(\underbrace{-\frac{f_{x_j}(\eta)}{f_{x_i}(\eta)}}_{i^{\text{th}} \text{ coordinate}}, \underbrace{1}_{j^{\text{th}} \text{ coordinate}}, 0, \ldots, 0 \Bigg) \in \mathbb{R}^n$$

and

$$P\left(\eta, u^j(\eta)\right) := \operatorname{span}\left\{ \nabla f(\eta), u^j(\eta) \right\} + \eta,$$

where $\operatorname{span}\left\{ \nabla f(\eta), u^j(\eta) \right\}$ means the space generated by the (linearly independent) vectors $\nabla f(\eta)$ and $u^j(\eta)$. So $P\left(\eta, u^j(\eta)\right)$ is, in fact, a 2-dimensional plane in $\mathbb{R}^n$ (it will simply be called a plane).

Directional curvature: definitions

Now I will present some directional notions, based on those presented at the beginning. To simplify the notation, we don't refer the unique (normal vector) ξ^*.

Definitions

- The 2-dimensional modulus of strict convexity of F at ξ in the direction of $u^j(\xi)$ is given by

$$\widehat{\mathfrak{C}}_F\left(r, \xi, u^j(\xi)\right) = \inf\left\{\langle \xi - \eta, \xi^* \rangle : \eta \in F \cap P\left(\xi, u^j(\xi)\right), \|\xi - \eta\| \geq r\right\}, \ r > 0.$$

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- The 2-dimensional curvature of F at ξ in the direction of $\mathfrak{u}^j(\xi)$ is given by

$$\hat{\varkappa}_F\left(\xi,\mathfrak{u}^j(\xi)\right)=\liminf_{\substack{(r,\eta)\to(0^+,\xi)\\ \eta\in\partial F}}\frac{2\,\widehat{\mathfrak{C}}_F\left(r,\eta,\mathfrak{u}^j(\eta)\right)}{\|\xi^*\|\,r^2}.$$

Directional curvature: results

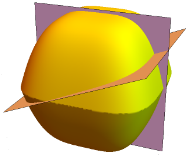
Proposition Denoting by $\hat{\varkappa}_F(\xi)$ our initial curvature, $\hat{\varkappa}_F(\xi, \xi^*)$, we have

$$\hat{\varkappa}_F(\xi, u(\xi)) \geq \hat{\varkappa}_F(\xi), \qquad \forall u(\xi) \in \mathbf{T}_F(\xi) \setminus \{\mathbf{0}\}. \tag{1}$$

Furthermore,

(i) if $\hat{\varkappa}_F(\xi, u(\xi)) = 0$ for some $u(\xi) \in \mathbf{T}_F(\xi) \setminus \{\mathbf{0}\}$, then $\hat{\varkappa}_F(\xi) = 0$;

(ii) the equality holds at (1) when $n = 2$.



Directional curvature: results

Theorem (The main result)

We have

$$\hat{\varkappa}_F\left(\xi, u^j(\xi)\right) = \frac{1}{\left\|\nabla f(\xi)\right\| \left\|u^j(\xi)\right\|^2} \left\langle \nabla^2 f(\xi)\, u^j(\xi), u^j(\xi) \right\rangle, \quad j \in I \setminus \{i\}, \qquad (2)$$

where $\nabla^2 f(\xi)$ *means the Hessian matrix, i.e.,* $\left(\nabla^2 f(\xi)\right)_{r,s} = \frac{\partial^2 f}{\partial x_r \partial x_s}(\xi)$, $r, s \in I$.

Corollary (1)

We have (2) for all $u(\xi) \in \mathbf{T}_F(\xi) \setminus \{\mathbf{0}\}$.

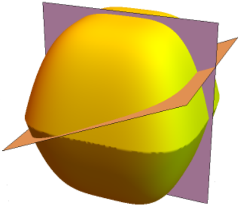
Directional curvature: results

Corollary (2)

We have

$$\hat{\varkappa}_F\left(\xi, u^j\left(\xi\right)\right) = \frac{f_{x_i x_i}\left(\xi\right) f_{x_j}^2\left(\xi\right) - 2 f_{x_i}\left(\xi\right) f_{x_j}\left(\xi\right) f_{x_i x_j}\left(\xi\right) + f_{x_j x_j}\left(\xi\right) f_{x_i}^2\left(\xi\right)}{\left\|\nabla f\left(\xi\right)\right\| \left(f_{x_i}^2\left(\xi\right) + f_{x_j}^2\left(\xi\right)\right)}.$$

Then I will compare the previous formula with the existing ones for curves in $\mathbb{R}^n$ generated by the intersection of $n-1$ implicit hypersurfaces.



Relation with the usual curvature formula for implicit space curves (n=2)

Near $\xi \in \partial F$, and fixed $u(\xi) \in \mathbf{T}_F(\xi) = \operatorname{span}\{(-f_{x_2}(\xi), f_{x_1}(\xi))\}$ with $u(\xi) \neq \mathbf{0}$, the curve $\partial F \cap P(\xi, u(\xi))$ is defined by the equation $\boxed{f(\eta) = 0}$. Therefore, by [G] (R. Goldman, Curvature Formulas for Implicit Curves and Surfaces, 2005),

$$
\begin{aligned}
k_G(\xi) &:= \frac{\left|\operatorname{Tan}(F)(\xi) * \nabla(\operatorname{Tan}(F))(\xi) * \nabla(F)(\xi)^{\mathrm{T}}\right|}{\|\nabla(F)\|^3} \\
&= \frac{\left|\begin{bmatrix} -f_{x_2}(\xi) & f_{x_1}(\xi) \end{bmatrix} \begin{bmatrix} f_{x_1x_1}(\xi) & f_{x_2x_1}(\xi) \\ f_{x_1x_2}(\xi) & f_{x_2x_2}(\xi) \end{bmatrix} \begin{bmatrix} -f_{x_2}(\xi) \\ f_{x_1}(\xi) \end{bmatrix}\right|}{\left(f_{x_1}^2(\xi) + f_{x_2}^2(\xi)\right)^{\frac{3}{2}}} \\
&= \frac{\left|f_{x_1}^2(\xi) f_{x_2x_2}(\xi) - 2 f_{x_2}(\xi) f_{x_1}(\xi) f_{x_1x_2}(\xi) + f_{x_2}^2(\xi) f_{x_1x_1}(\xi)\right|}{\left(f_{x_1}^2(\xi) + f_{x_2}^2(\xi)\right)^{\frac{3}{2}}} \\
&= \hat{\varkappa}_F(\xi, u(\xi)).
\end{aligned}
$$

Relation with the usual curvature formula for implicit space curves ($n \geq 3$)

Near $\xi \in \partial F$, for a fixed $\omega^j(\xi)$ with $j \in I \setminus \{i\}$, the curve $\partial F \cap P\left(\xi, \omega^j(\xi)\right)$ is defined by the $n-1$ equations:

$$\boxed{f(\eta) = 0, \quad p_{k_1 \xi}(\eta) = 0, ..., p_{k_{n-2}\xi}(\eta) = 0,}$$

$k_1, ..., k_{n-2} \in I \setminus \{i, j\}$, where, for $\eta = (\eta_1, ..., \eta_n) \in \mathbb{R}^n$,

$$p_{k\xi}(\eta) := \eta_k - \xi_k + a_{ki}(\xi)(\eta_i - \xi_i) + a_{kj}(\xi)(\eta_j - \xi_j)$$

and

$$a_{kl}(\xi) := -\frac{f_{x_l}(\xi) f_{x_k}(\xi)}{f_{x_i}^2(\xi) + f_{x_j}^2(\xi)}, \quad l \in \{i, j\}.$$

Relation with the usual curvature formula for implicit space curves ($n \geq 3$)

From [G], and after many calculations, we get:

$$
\begin{aligned}
k_G(\xi) &:= \frac{\left\| \left(\operatorname{Tan}(F)(\xi) * \nabla \left(\operatorname{Tan}(F) \right)(\xi) \right) \wedge \operatorname{Tan}(F)(\xi) \right\|}{\left\| \operatorname{Tan}(F)(\xi) \right\|^3} \\
&= \frac{\left| f_{x_i x_i}(\xi) f_{x_j}^2(\xi) - 2 f_{x_i}(\xi) f_{x_j}(\xi) f_{x_i x_j}(\xi) + f_{x_j x_j}(\xi) f_{x_i}^2(\xi) \right|}{\left\| \nabla f(\xi) \right\| \left(f_{x_i}^2(\xi) + f_{x_j}^2(\xi) \right)} \\
&= \hat{\varkappa}_F \left(\xi, u^j(\xi) \right).
\end{aligned}
$$

(Where $\operatorname{Tan}(F)(\xi)$ means the row matrix $\operatorname{Tan}\left(f, p_{k_1 \xi}, \ldots, p_{k_{n-2} \xi} \right)(\xi) := \begin{vmatrix} e \\ \nabla f(\xi) \\ \nabla p_{k_1 \xi}(\xi) \\ \vdots \\ \nabla p_{k_{n-2} \xi}(\xi) \end{vmatrix}$

and $\wedge$ the generalization to the cross product from $\mathbb{R}^3$ to $\mathbb{R}^n$.)

Directional curvature: conclusions

Given a convex body $F \subset \mathbb{R}^n$, $n \geq 2$, and a point $\xi \in \partial F$ both checking our conditions.

1. For any $u(\xi) \in \mathbf{T}_F(\xi) \setminus \{\mathbf{0}\}$ the directional curvature $\hat{\varkappa}_F(\xi, u(\xi))$ can be easily calculated using the formula:

$$\hat{\varkappa}_F(\xi, u(\xi)) = \frac{1}{\|\nabla f(\xi)\| \|u(\xi)\|^2} \left\langle \nabla^2 f(\xi) u(\xi), u(\xi) \right\rangle.$$

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2. For the principal directions $u^j(\xi)$, $j \in I \setminus \{i\}$, the directional curvature $\hat{\varkappa}_F\left(\xi, u^j(\xi)\right)$ can also be calculated using Goldman's formula $k_G(\xi)$, but it takes more work.

Directional derivative

Theorem

Let $u(\xi) \in \mathbf{T}_F(\xi) \setminus \{\mathbf{0}\}$. *Assume that* $\widehat{\mathfrak{C}}_F(r, \xi, u(\xi)) > 0$, *for all* $r > 0$. *Then the directional derivative of* $\rho_{F^\circ}(\cdot)$ *at* ξ^* *in the direction of* $u(\xi)$ *exists and is given by*

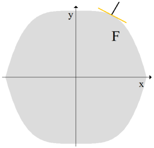
$$\rho'_{F^\circ}(\xi^*, u(\xi)) = \langle \xi, u(\xi) \rangle .$$

Note In [GP] we showed that:

$$\widehat{\mathfrak{C}}_F(r, \xi, \xi^*) > 0 \text{ for all } r > 0 \Longrightarrow \nabla \rho_{F^\circ}(\xi^*) = \xi.$$

Example 1

$$F = \left\{ (x_1, x_2) \in \mathbb{R}^2 : |x_2| \leq 1 - x_1^4,\ -1 \leq x_1 \leq 1 \right\}.$$



Fix $\xi = (\xi_1, \xi_2) \in \partial F$

- if $\xi_1, \xi_2 > 0$ we have $\mathbf{T}_F(\xi) = \operatorname{span}\left\{ \left(1, -4\xi_1^3\right) \right\}$,

$$\hat{\varkappa}_F(\xi) = \hat{\varkappa}_F(\xi, u(\xi)) = \frac{12\xi_1^2}{\left(16\xi_1^6 + 1\right)^{\frac{3}{2}}}, \quad u(\xi) \in \mathbf{T}_F(\xi) \setminus \{(0,0)\},$$

whereas in [GP] we had only obtained the inequalities

$$\frac{12\xi_1^2}{\sqrt{1 + 16\xi_1^6}\ \Sigma(\xi_1)} \leq \hat{\varkappa}_F(\xi) \leq \frac{12\xi_1^2}{\sqrt{1 + 16\xi_1^6}}, \quad \Sigma(\xi_1) := 1 + \left(\sum_{k=0}^{3} |\xi_1|^k \right)^2;$$

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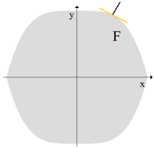

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- if $\xi_{1,}\xi_2 > 0$ we have $\mathbf{T}_F(\xi) = \text{span}\left\{ \left(1, -4\xi_1^3\right) \right\}$,

$$\hat{\varkappa}_F(\xi) = \hat{\varkappa}_F(\xi, u(\xi)) = \frac{12\xi_1^2}{\left(16\xi_1^6 + 1\right)^{\frac{3}{2}}}, \quad u(\xi) \in \mathbf{T}_F(\xi) \setminus \{(0,0)\},$$

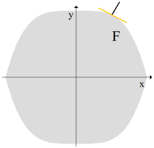
  whereas in [GP] we had only obtained the inequalities

$$\frac{12\xi_1^2}{\sqrt{1 + 16\xi_1^6 \, \Sigma(\xi_1)}} \leq \hat{\varkappa}_F(\xi) \leq \frac{12\xi_1^2}{\sqrt{1 + 16\xi_1^6}}, \quad \Sigma(\xi_1) := 1 + \left(\sum_{k=0}^{3} |\xi_1|^k \right)^2;$$

- if $\xi = (0, \pm 1)$ we have $\hat{\varkappa}_F(\xi) = \hat{\varkappa}_F(\xi, u(\xi)) = 0$;

Example 1

$$F = \left\{ (x_1, x_2) \in \mathbb{R}^2 : |x_2| \le 1 - x_1^4,\ -1 \le x_1 \le 1 \right\}.$$



Fix $\xi = (\xi_1, \xi_2) \in \partial F$

- if $\xi_1, \xi_2 > 0$ we have $\mathbf{T}_F(\xi) = \text{span}\left\{ \left(1, -4\xi_1^3\right) \right\}$,

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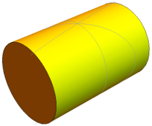
$$\frac{12\xi_1^2}{\sqrt{1 + 16\xi_1^6}\ \Sigma(\xi_1)} \le \hat{\varkappa}_F(\xi) \le \frac{12\xi_1^2}{\sqrt{1 + 16\xi_1^6}}, \quad \Sigma(\xi_1) := 1 + \left(\sum_{k=0}^{3} |\xi_1|^k \right)^2;$$

- if $\xi = (0, \pm 1)$ we have $\hat{\varkappa}_F(\xi) = \hat{\varkappa}_F(\xi, u(\xi)) = 0$;
- if $\xi = (\pm 1, 0)$ we can't calculate the directional curvature because there isn't a $\mathcal{C}^2$ function f checking our conditions.

Example 2

$$F_{a,b} = \left\{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2 + x_3^2 \leq a^2,\ |x_2| \leq b\right\}, \quad a, b \in \mathbb{R}^+.$$



Fix $\xi = (\xi_1, \xi_2, \xi_3) \in \partial F_{a,b}$

- if $\xi_1^2 + \xi_3^2 = a^2$ and $|\xi_2| < b$ we have $\mathbf{T}_{F_{a,b}}(\xi) = \text{span}\left\{\underbrace{(0,1,0)}_{u^2(\xi)}, \underbrace{(-\xi_3, 0, \xi_1)}_{u^j(\xi),\ j\in\{1,3\}\setminus\{i\}}\right\}$

  and

$$\hat{\varkappa}_{F_{a,b}}(\xi, u(\xi)) = \frac{\beta^2 a}{a^2\beta^2 + \xi_1^2\alpha^2} \in \left[0, \frac{1}{a}\right]$$

  for every $u(\xi) = \alpha u^2(\xi) + \beta u^j(\xi)$, with $\alpha, \beta \in \mathbb{R}$ not simultaneously null;

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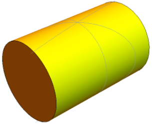

Fix $\xi = (\xi_1, \xi_2, \xi_3) \in \partial F_{a,b}$

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  for every $u(\xi) = \alpha u^2(\xi) + \beta u^j(\xi)$, with $\alpha, \beta \in \mathbb{R}$ not simultaneously null;

- if $\xi_1^2 + \xi_3^2 < a^2$ and $\xi_2 = b$ (the case $\xi_2 = -b$ is analogous) we have

  $$\hat{\varkappa}_{F_{a,b}}(\xi, u(\xi)) = 0,$$

  for every $u(\xi) \in \mathbf{T}_{F_{a,b}}(\xi) = \text{span}\{(1,0,0), (0,0,1)\}$, with $u(\xi) \neq (0,0,0)$.

Example 3

$$F = \left\{ (x_1, ..., x_n) \in \mathbb{R}^n : \sum_{t=1}^{n} x_t^2 \leq R^2 \right\}, \quad R > 0.$$

Near any $\xi \in \partial F$ we have $f(x) = \sum_{t=1}^{n} x_t^2 - R^2$. Consequently, fixed the first $i \in I$ such that $f_{x_i}(\xi) = 2\xi_i \neq 0$, we have

$$\mathbf{T}_F(\xi) = \operatorname{span} \left\{ \bigcup_{j \in I \setminus \{i\}} u^j(\xi) \right\},$$

where $u^j(\xi)$ is the vector with 1 in the jth coordinate, $\frac{\xi_j}{\xi_i}$ in the ith coordinate and 0 in the others. Therefore

$$\hat{\varkappa}_F\left(\xi, u^j(\xi)\right) = \frac{\left(\left(\frac{\xi_j}{\xi_i}\right)^2 + 1\right)^2}{\|\xi\| \left\|u^j(\xi)\right\|^2} = \frac{1}{R},$$

as we expected.

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Thank you very much!