Rotational bounds for homeomorphisms with integrable distortion and Hölder continuous inverse

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Introduction

Modulus of continuity of inverse maps

Mappings of finite distortion

A homeomorphism $f : \mathbb{C} \to \mathbb{C}$ is a map of finite distortion if:

- $f \in W^{1,1}_{loc}$
- $\det(Df) = J(\cdot, f) \in L^1_{loc}$
- There exists a measurable function $\mathbb{K} : \mathbb{C} \to [1, +\infty)$ such that $|Df(z)|^2 \leq \mathbb{K}(z, f) \cdot J(z, f)$ at almost every z.

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Modulus of continuity (Assumption $f(0) = 0$)

- (AHLFORS) f K-QC: $|f(z)| \geq \frac{1}{c_K}|z|^K$; $K = \|\mathbb{K}\|_\infty$.
- (HERRON-KOSKELA) $e^{\mathbb{K}(\cdot, f)} \in L^p_{loc}$ $(p > 0)$:
  $|f(z)| \geq e^{-\frac{c_{f,p}}{p}\log^2(\frac{1}{|z|})}$.
- (KOSKELA-TAKKINEN) $\mathbb{K} \in L^p_{loc}$ $(p > 1)$: $|f(z)| \geq e^{-c_{f,p}|z|^{-\frac{2}{p}}}$.

Rotational properties of planar maps

Given $f : \mathbb{C} \to \mathbb{C}$ with $f(0) = 0$ and $f(1) = 1$, we are interested in the growth of $|\arg(f(r))|$ as $r \to 0$.

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Astala-Iwaniec-Prause-Saksman (2014)

If f is K-quasiconformal, then

$$|\arg(f(r))| \leq \frac{1}{2}\left(K - \frac{1}{K}\right)\log\left(\frac{1}{r}\right) + c_K, \qquad \forall r \in (0,1).$$

Rotational properties of planar maps

Hitruhin (2018)

- If $e^{\mathbb{K}(\cdot,f)} \in L^p_{loc}$ for some $p > 0$, then

$$|\arg(f(z))| \le \frac{c}{p} \log^2\left(\frac{1}{|z|}\right), \qquad \text{for } |z| \text{ small enough.}$$

- When $\mathbb{K}(\cdot,f) \in L^p_{loc}$ for some $p > 1$,

$$|\arg(f(z))| \le \frac{c}{|z|^{\frac{2}{p}}}.$$

- If $\mathbb{K}(\cdot,f) \in L^1_{loc}$, then

$$\limsup_{|z|\to 0} |z|^2 \, |\arg(f(z))| = 0.$$

Euler equation in the plane

Planar Euler equation in vorticity form

Euler equation

$$EE : \begin{cases} \omega_t + v \cdot \nabla \omega = 0, \\ \omega(0, \cdot) = \omega_0, \\ v = K * \omega \end{cases}$$

- $v(t, \cdot) : \mathbb{R}^2 \to \mathbb{R}^2$ velocity field
- $\omega(t, \cdot) : \mathbb{R}^2 \to \mathbb{R}$ vorticity
- $K=$ Convolution Kernel

$$K(z) = K(x, y) = \frac{iz}{2\pi |z|^2} \equiv \frac{1}{2\pi} \frac{(-y, x)}{x^2 + y^2}$$

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Biot-Savart Law

$$v = K * \omega \quad \Longleftrightarrow \quad \begin{cases} \mathrm{div}(v) = 0 \\ \mathrm{curl}(v) = \omega \end{cases} \quad \Longleftrightarrow \quad \partial_z v = \frac{i\omega}{2}$$

Euler Flows are Hölder

Yudovich (1963)

If $\omega_0 \in L^\infty_c$, then there exists an unique solution $\omega \in L^\infty(0, T; L^\infty)$.

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If ω is an Yudovich solution, then $\omega(t, \cdot) \in L^\infty$

$$\Rightarrow \partial_z v = \frac{i\omega}{2} \in L^\infty$$
$$\Rightarrow \partial_{\bar{z}} v \in BMO$$
$$\Rightarrow v \text{ is } Zygmund$$
$$\Rightarrow v \text{ is } Lip - Log$$
$$\Rightarrow v \text{ has flow } X_t \in C^{\alpha(t)}$$

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\end{aligned}$$

Bahouri-Chemin (1993)

$\alpha(t) \leq e^{-t\,\|\omega_0\|_\infty}$

Euler Flows are Sobolev

Clop-Jylhä (2019)

If $\omega \in L^\infty(L^\infty)$ is an Yudovich solution, and $v = K * \omega$, then

$$X_t \in W^{1,p}_{loc}$$

for $1 < p < \dfrac{2}{t\|\omega_0\|_\infty}$

Improved rotation

Motivation for improving Hitruhin's result

- For small times $t > 0$, Euler flows X_t are mappings of finite distortion with L^p_{loc} distortion.

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- Wolibner 1933: Euler flows have Hölder continuous inverse.

Improved rotation for Euler flows

Clop-Hitruhin-S. 2021

Let $p, \alpha > 1$. If

- $f : \mathbb{C} \to \mathbb{C}$ homeomorphism of finite distortion with $\mathbb{K}(\cdot, f) \in L^p_{loc}$
- $f(0) = 0$, $f(1) = 1$
- $|f(x) - f(y)| \geq |x - y|^\alpha$, whenever $|x - y|$ is small

then

$$|\arg(f(z))| \leq C\sqrt{\alpha}|z|^{-\frac{1}{p}} \log^{\frac{1}{2}}\left(\frac{1}{|z|}\right).$$

Improved rotation for Euler flows

Corollary

Given $\omega_0 \in L^\infty(\mathbb{C};\mathbb{C})$, let X_t be Euler flow. Then there is a constant $C > 0$ such that

$$\left|\arg\left(\frac{X_t(z) - X_t(0)}{X_t(1) - X_t(0)}\right)\right| \leq C \,\log^{\frac{1}{2}}\left(\frac{1}{|z|}\right) |z|^{-t\|\omega_0\|_\infty} \exp\left(Ct\|\omega_0\|_\infty\right)$$

if both $|z|$ and $t > 0$ are small enough.

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Given $\omega_0 \in L^\infty(\mathbb{C};\mathbb{C})$, let X_t be Euler flow. Then there is a constant $C > 0$ such that

$$\left|\arg\left(\frac{X_t(z) - X_t(0)}{X_t(1) - X_t(0)}\right)\right| \leq C \, \log^{\frac{1}{2}}\left(\frac{1}{|z|}\right) \, |z|^{-t\|\omega_0\|_\infty} \, \exp\left(Ct\|\omega_0\|_\infty\right)$$

if both $|z|$ and $t > 0$ are small enough.

The curve $X_{t_0}([\frac{1}{n}, 1])$ cannot wind around $X_{t_0}(0)$ more than a multiple of

$$n^{t_0\|\omega_0\|_\infty} \, (\log n)^{\frac{1}{2}} \, e^{Ct_0\|\omega_0\|_\infty}$$

times.

Optimal result

Theorem

Given an increasing, onto homeomorphism $h : [0, +\infty) \to [0, +\infty)$, and a real number $p > 1$,
there exists a homeomorphism $g : \mathbb{C} \to \mathbb{C}$
with the following properties:

- g homeomorphism of finite distortion with $\mathbb{K}(\cdot, g) \in L^p_{loc}$
- $g(0) = 0$, $g(1) = 1$
- If $\alpha > \frac{3p}{p-1}$, then $|g(x) - g(y)| \geq C|x - y|^\alpha$ whenever $|x - y| < 1$
- There exists a decreasing sequence $\{r_n\}$, with limit $r_n \to 0^+$ as $n \to \infty$, for which

$$|\arg(g(r_n))| \geq r_n^{-\frac{1}{p}} \log^{\frac{1}{2}} \left(\frac{1}{r_n}\right) h(r_n).$$

Borderline case

Hitruhin-S. 2021

Let $\alpha \geq 1$. If

- $f : \mathbb{C} \to \mathbb{C}$ homeomorphism of finite distortion with $\mathbb{K}(\cdot, f) \in L^1_{loc}$
- $f(0) = 0$, $f(1) = 1$
- $|f(x) - f(y)| \geq |x - y|^\alpha$, whenever $|x - y|$ is small

then

$$\limsup_{|z| \to 0} \frac{|z|}{\sqrt{\log\left(\frac{1}{|z|}\right)}} |\arg(f(z))| = 0.$$

Optimality

Theorem

Given an increasing, onto homeomorphism $h : [0, +\infty) \to [0, +\infty)$, there exists a homeomorphism $g : \mathbb{C} \to \mathbb{C}$ with the following properties:

- g homeomorphism of finite distortion with $\mathbb{K}(\cdot, g) \in L^1_{loc}$
- $g(0) = 0$, $g(1) = 1$
- If $\alpha > 6$, then $|g(x) - g(y)| \geq C|x - y|^\alpha$ whenever $|x - y| < 1$
- There exists a decreasing sequence $\{r_n\}$, with limit $r_n \to 0^+$ as $n \to \infty$, for which

$$|\arg(g(r_n))| \geq \frac{h(r_n)}{r_n} \log^{\frac{1}{2}} \left(\frac{1}{r_n} \right).$$

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- If $\alpha > 6$, then $|g(x) - g(y)| \geq C|x - y|^\alpha$ whenever $|x - y| < 1$
- There exists a decreasing sequence $\{r_n\}$, with limit $r_n \to 0^+$ as $n \to \infty$, for which

$$|\arg(g(r_n))| \geq \frac{h(r_n)}{r_n} \log^{\frac{1}{2}}\left(\frac{1}{r_n}\right).$$

Remark: The method used in the proof of this result certainly improves the lower bound of α in the optimal result for $p > 1$ case from $3p/(p-1)$ to $3(p+1)/p$.

Growth condition

Hitruhin-S. 2021

Let $f : \mathbb{C} \to \mathbb{C}$ be a homeomorphism of finite distortion such that $f(0) = 0$, $f(1) = 1$, and assume that $\mathbb{K}(\cdot, f) \in L^p_{loc}$; $p > 1$. Then

$$|\arg(f(z))| \leq C\,|z|^{-\frac{1}{p}} \log^{\frac{1}{2}} \left(\frac{1}{\min\limits_{|\omega|=|z|} |f(\omega)|} \right) \quad \text{when } |z| \text{ is small.}$$

Furthermore, if we assume that $\mathbb{K}(\cdot, f) \in L^1_{loc}$, then

$$\limsup_{|z|\to 0} \frac{|z|}{\sqrt{\log\left(\frac{1}{\min\limits_{|\omega|=|z|} |f(\omega)|} \right)}} |\arg(f(z))| = 0.$$

Optimality result

Theorem

Given φ radially increasing homeomorphism with $\mathbb{K}(\cdot,\varphi) \in L^p_{loc}$, $p \geq 1$, such that, when $|z|$ is small,

$$e^{-m_{\varphi,p}(|z|)|z|^{-\frac{2}{p}}} \leq |\varphi(z)| < |z|^4,$$

where $m_{\varphi,p} : \mathbb{R} \to \mathbb{R}$ increasing continuous with $m(r) \to 0$ as $r \to 0$, and $h : [0,+\infty) \to [0,+\infty)$, we can find a radial homeomorphism $g : \mathbb{C} \to \mathbb{C}$;

- g homeomorphism with $\mathbb{K}(\cdot, g) \in L^p_{loc}$; $g(0) = 0$, $g(1) = 1$.
- There exists a decreasing sequence $\{r_n\}$, such that

$$|g(r_n)| = |\varphi(r_n)|$$

and

$$|\arg(g(r_n))| \geq r_n^{-\frac{1}{p}} \log^{\frac{1}{2}}\left(\frac{1}{|g(r_n)|}\right) h(r_n).$$

Thanks for your attention