Rogers-Shephard type inequalities for the lattice point enumerator

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(joint work with David Alonso-Gutiérrez and Eduardo Lucas)

Geometry, Analysis and Convexity

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The Brunn-Minkowski inequality ((1/n)-concave form)

Let $K, L \subset \mathbb{R}^n$ be non-empty compact sets. Then, for all $\lambda \in (0,1)$,

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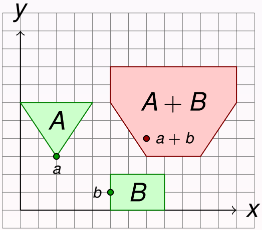
Here $\mathrm{vol}(\cdot)$ is the Lebesgue measure and $A + B = \{a + b : a \in A, b \in B\}$ denotes the Minkowski sum of A and B, and $rA = \{ra : a \in A\}$ for any $r \geq 0$.



Preliminaries

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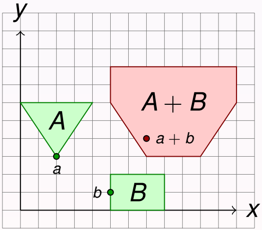

$\rightarrow$ It yields the isoperimetric inequality in a few lines: *Among all sets with a fixed surface area measure, Euclidean balls maximize the volume.*

The Rogers-Shephard inequality

The Rogers-Shephard inequality, originally proven in 1957, provides us with an upper bound for the volume of $K - K$:

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This relation for $K - K$ can be generalized to the Minkowski addition of two convex bodies as follows:

Let $K, L \in \mathcal{K}^n$ be convex bodies. Then

$$\mathrm{vol}(K + L)\mathrm{vol}\big(K \cap (-L)\big) \leq \binom{2n}{n} \mathrm{vol}(K)\mathrm{vol}(L).$$

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In 1958 Rogers and Shephard gave the following lower bound for the volume of K in terms of the volumes of a projection and a maximal section of K:

Section/Projection Rogers-Shephard's inequality

Let $k \in \{1, \ldots, n-1\}$ and $H \in \mathcal{L}_k^n$. Let $K \in \mathcal{K}^n$ be a convex body. Then

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The Rogers-Shephard inequality for the difference body can be derived from the latter result for sections & projections

A related result: Berwald's inequality

Berwald's inequality

Let $K \in \mathcal{K}^n$ be a convex body with $\dim K = n$ and let $f : K \longrightarrow \mathbb{R}_{\geq 0}$ be a concave function. Then, for any $0 < p < q$,

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- Rogers-Shephard's inequality for $K - K$ $\rightsquigarrow$ consider the function $f : K - K \longrightarrow \mathbb{R}_{\geq 0}$ defined by

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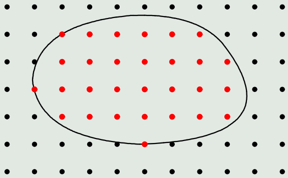

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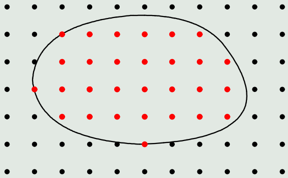

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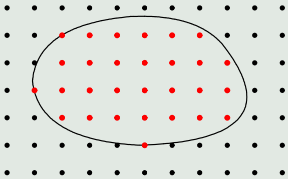

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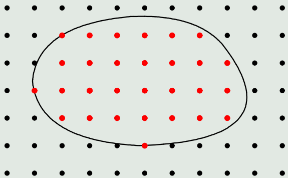

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- Berg's thesis (2018): Brunn's inequality.
- Freyer and Henk (2020): Meyer's inequality, reverse Meyer's inequality and reverse Loomis-Whitney's inequality.

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Let $K, L \subset \mathbb{R}^n$ be non-empty bounded sets. Then, for all $\lambda \in (0,1)$,

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- Jointly with Iglesias and Lucas (2022), and exploiting a functional approach, the latter result can be extended to the case of arbitrary $t, s \geq 0$ (instead of $1-\lambda$ and λ) by replacing $(-1,1)^n$ by $(-1, \lceil t+s \rceil)^n$.

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Let $K, L \subset \mathbb{R}^n$ be non-empty bounded sets. Then

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R-S type inequalities for the lattice point enumerator?

Taking into account the strong connection between the Brunn-Minkowski inequality and the Rogers-Shephard inequality, it is natural to wonder about getting a discrete version of the latter, for $\mathrm{G}_n(\cdot)$.

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Theorem (Gardner, Gronchi (2001))

Let $P \subset \mathbb{R}^2$ be a convex polygon with integer vertices. Then

$$\mathrm{G}_2(P - P) \leq 6\mathrm{G}_2(P) - \mathrm{b}(P) - 5, \qquad (1)$$

where $\mathrm{b}(P)$ denotes the number of integer points in the boundary of P.

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Furthermore, as pointed out by Freyer&Henk (2021), there is neither a possible extension of (1) in dimension $n \geq 3$ nor even a hope to obtain $\mathrm{G}_n(K - K) \leq c_n\mathrm{G}_n(K)$ for some constant $c_n > 0$, for $n \geq 3$.

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Theorem (Alonso-Gutiérrez, Lucas, Y. N. (2022+))

Let $k \in \{1, \dots, n-1\}$ and $H = \mathrm{lin}\{\mathrm{e}_1, \dots, \mathrm{e}_k\} \in \mathcal{L}^n_k$. Let $K \subset \mathbb{R}^n$ be a non-empty convex bounded set. Then

$$\mathrm{G}_{n-k}(P_{H^\perp} K)\mathrm{G}_k(K \cap H) \leq \binom{n}{k}\mathrm{G}_n\big(K + (-1, 1)^n\big).$$

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Theorem (Alonso-Gutiérrez, Lucas, Y. N. (2022+))

Let $K \subset \mathbb{R}^n$ be a convex bounded set containing the origin and let $f : K \longrightarrow \mathbb{R}_{\geq 0}$ be a concave function with $f(0) = |f|_\infty$. Then, for any $0 < p < q$,

$$\left(\frac{\binom{n+q}{n}}{\mathrm{G}_n(K)} \sum_{x \in K \cap \mathbb{Z}^n} f^q(x) \right)^{1/q} \leq \left(\frac{\binom{n+p}{n}}{\mathrm{G}_n(K)} \sum_{x \in (K + (-1,1)^n) \cap \mathbb{Z}^n} \left(f^{\diamond}\right)^p(x) \right)^{1/p}.$$

A Berwald type inequality for the lattice point enumerator

Let $K \in \mathcal{K}^n$ be a convex body with $\dim K = n$ and let $f : K \longrightarrow \mathbb{R}_{\geq 0}$ be a concave function. Then, for any $0 < p < q$,

$$\left(\frac{\binom{n+q}{n}}{\mathrm{vol}(K)} \int_K f^q(x)\,\mathrm{d}x \right)^{1/q} \leq \left(\frac{\binom{n+p}{n}}{\mathrm{vol}(K)} \int_K f^p(x)\,\mathrm{d}x \right)^{1/p}.$$

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For a function ϕ defined on K we denote by $\phi^\diamond := \overline{\phi} \star \chi_{(-1,1)^n}$, where

$$\overline{\phi}(x) := \begin{cases} \phi(x) & \text{if } x \in K, \\ 0 & \text{otherwise.} \end{cases}$$

From this, arguing as in the continuous setting (for the above-mentioned functions and values of p, and letting $q \to \infty$), we get the following:

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Corollary (Alonso-Gutiérrez, Lucas, Y. N. (2022+))

Let $K \subset \mathbb{R}^n$ be a convex bounded set containing the origin. Then

$$\mathrm{G}_n(K-K)\mathrm{vol}(K) \leq \binom{2n}{n} \sum_{x\in(K-K+(-1,1)^n)\cap\mathbb{Z}^n} \sup_{z\in(-1,1)^n} \mathrm{vol}\Big(K\cap\big((x+z)+K\big)\Big).$$

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$$\mathrm{G}_n(K+L)\mathrm{vol}\big(K\cap(-L)\big) \leq \binom{2n}{n} \sum_{x \in (K+L+(-1,1)^n)\cap\mathbb{Z}^n} \sup_{z\in(-1,1)^n} \mathrm{vol}\Big(K \cap \big((x+z)-L\big)\Big).$$

Other R-S type inequalities for the lattice point enumerator

From this, arguing as in the continuous setting (for the above-mentioned functions and values of p, and letting $q \to \infty$), we get the following:

Corollary (Alonso-Gutiérrez, Lucas, Y. N. (2022+))

Let $K \subset \mathbb{R}^n$ be a convex bounded set containing the origin. Then

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Corollary (Alonso-Gutiérrez, Lucas, Y. N. (2022+))

Let $k \in \{1, \dots, n-1\}$ and $H \in \mathcal{L}^n_k$. Let $K \subset \mathbb{R}^n$ be a convex bounded set containing the origin. Then

$$\mathrm{G}_{n-k}(P_{H^\perp}K)\mathrm{vol}_k(K \cap H) \leq \binom{n}{k} \sum_{x \in (P_{H^\perp}K + \mathrm{C}_{H^\perp})\cap\mathbb{Z}^n} \sup_{z \in \mathrm{C}_{H^\perp}} \mathrm{vol}_k\Big(K \cap \big((x+z)+H\big)\Big).$$

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Here we also prove that these new discrete analogues for $\mathrm{G}_n(\cdot)$ imply the corresponding results involving the volume $\mathrm{vol}(\cdot)$.

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$$\lim_{r\to\infty} \frac{\mathrm{G}_n(rK)}{\mathrm{vol}(rK)} = 1.$$

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- For the Berwald type inequalities, we exploit the following:

Given $K \in \mathcal{K}^n$ and a concave function $f : K \longrightarrow \mathbb{R}_{\geq 0}$, we have

$$\lim_{r\to\infty} \left[\frac{1}{r^n} \sum_{x \in (rK)\cap\mathbb{Z}^n} f\left(\frac{x}{r}\right) \right] = \int_K f(x)\,\mathrm{d}x.$$

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This is due to the fact that f is Riemann integrable (because it is concave on the convex set K, whose boundary has null measure).

Rogers-Shephard type inequalities for the lattice point enumerator

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Universidad de Murcia

(joint work with David Alonso-Gutiérrez and Eduardo Lucas)

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Thank you very much!!