Affine Minkowski valuations

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Outline

Notation & Definition

- $V = \mathbb{R}^n$, W real vector space
- $\mathrm{MVal}(V, W)$ space of continuous translation invariant Minkowski valuations $\mathcal{K}(V) \to \mathcal{K}(W)$

Definition (Ludwig '05)

$Z\colon \mathcal{K}(V) \to \mathcal{K}(W)$ is called *Minkowski valuation* if

$$Z(K) + Z(L) = Z(K \cap L) + Z(K \cup L),$$

whenever $K, L, K \cup L \in \mathcal{K}(V)$.

Z is

- *translation invariant* if $Z(K + t) = Z(K)$
- *even* if $Z(K) = Z(-K)$
- *homogeneous of degree k* if $Z(\lambda K) = \lambda^k Z(K)$, $\lambda \geq 0$

Representations of $\mathrm{SL}(V)$

- If $\mathrm{SL}(V)$ acts on W linearly, we say W is a representation of $\mathrm{SL}(V)$.
- W is *irreducible* if there is no nontrivial proper $\mathrm{SL}(V)$ invariant subspace.
- Z is $\mathrm{SL}(V)$ equivariant if

$$Z(\phi \cdot K) = \phi \cdot Z(K)$$

Examples: $\mathrm{SL}(V)$ acts on

- V by $g \cdot v := g(v)$.
- V^* by $g \cdot \xi := \xi \circ g^{-1}$.
- $\mathbb{R}$ by $g \cdot v := v$.

Exterior and Symmetric Power

$\mathrm{SL}(V)$ acts on $V^{\otimes k}$ by

$$g \cdot (v_1 \otimes \cdots \otimes v_k) := g(v_1) \otimes \cdots \otimes g(v_k).$$

$\bigwedge^k V, \mathrm{Sym}^k V \subset V^{\otimes k}$ irreducible subspaces.

- $\bigwedge^k V := \langle v_1 \wedge \cdots \wedge v_k : v_i \in V \rangle$, where

$$v_1 \wedge \cdots \wedge v_k := \frac{1}{k!} \sum_{\sigma \in S_k} \mathrm{sgn}(\sigma) \cdot v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(k)}$$

- $\mathrm{Sym}^k V := \langle v_1 \odot \cdots \odot v_k : v_i \in V \rangle$, where

$$v_1 \odot \cdots \odot v_k := \frac{1}{k!} \sum_{\sigma \in S_k} v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(k)}.$$

Examples

- For fixed closed intervals $I_1, I_2 \subset \mathbb{R}$ we have

$$J \colon \mathcal{K}(V) \to \mathcal{K}(\mathbb{R}), \quad K \mapsto I_1 + \mathrm{vol}(K) \cdot I_2.$$

- Difference body

$$D \colon \mathcal{K}(V) \to \mathcal{K}(V), \quad K \mapsto K + (-K).$$

- Projection body $\Pi \colon \mathcal{K}(V) \to \mathcal{K}(V^*)$ defined by the support function

$$h_{\Pi K}(u) = \tfrac{n}{2} V_n(K[n-1], [-u, u]),$$

where V_n denotes the mixed volume. Or equivalently, if V^* is identified with V via the euclidean structure:

$$h_{\Pi K}(u) = \mathrm{vol}_{n-1}(\pi_{u^\perp}(K)), \quad u \in S^{n-1}.$$

All these examples are continuous, $\mathrm{SL}(V)$ equivariant, translation invariant Minkowski valuations.

Difference body vs. Projection body

Via $V^* = \bigwedge^{n-1} V$:

$$\Pi \colon \mathcal{K}(V) \to \mathcal{K}(\textstyle\bigwedge^{n-1} V).$$

Via $V = \bigwedge^1 V$:

$$D \colon \mathcal{K}(V) \to \mathcal{K}(\textstyle\bigwedge^1 V)$$

and

$$h_{DK}(u) = 2 \cdot \mathrm{vol}_1(\pi_{\langle u \rangle}(K)).$$

What about k-th projection bodies $\mathcal{K}(V) \to \mathcal{K}(\bigwedge^k V)$ for $1 < k < n-1$?

Question

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Are there more continuous, $\mathrm{SL}(V)$ equivariant Minkowski valuations?

What is already known?

Theorem (Ludwig '05)

Let $n \geq 2$*. If* $Z \in \mathrm{MVal}(V, V)$ *(resp.* $Z \in \mathrm{MVal}(V, V^*)$*) is* $\mathrm{SL}(V)$ *equivariant, then* Z *is a multiple of the difference body (resp. projection body).*

Main result

Theorem (H.–Wannerer '22+)

Let W be an irreducible representation of $\mathrm{SL}(V)$ (of finite dimension). If $Z \in \mathrm{MVal}(V, W)$ is non-trivial and $\mathrm{SL}(V)$ equivariant, then W is isomorphic (as representation) to either V, V^* or $\mathbb{R}$.

Klain function

$\mathrm{Val}_k^{(+)}(V)$ space of continuous translation invariant (even) valuations $\mathcal{K}(V) \to \mathbb{R}$ homogeneous of degree k.

Theorem (Hadwiger '57)

$\varphi \in \mathrm{Val}_n(V)$ is a multiple of the volume.

Let $E \in \mathrm{Gr}_k(V)$. Hadwiger implies

$$\varphi \in \mathrm{Val}_k \Rightarrow \varphi|_{\mathcal{K}(E)} = c_E \cdot \mathrm{vol}_k.$$

The map

$$\mathrm{Kl}_\varphi \colon \mathrm{Gr}_k(V) \to \mathbb{R}, \quad E \mapsto c_E$$

is called *Klain function.*

Theorem (Klain '99)

$\varphi \in \mathrm{Val}_k^+(V)$ is uniquely determined by its Klain function.

Sketch of the proof

- It is enough to consider $Z \in \mathrm{MVal}_k^+(V, W)$, where $k \in \{0, \dots, n\}$.

Theorem

If $Z \in \mathrm{MVal}_k^+(V, W)$ is $\mathrm{SL}(V)$ equivariant it is uniquely determined by a convex body $\mathrm{Kl}_Z \subset W$. Moreover $\mathrm{Kl}_Z = Z(K)$, where K is any convex body in $\mathbb{R}^k \subset V$ with $vol_k(K) = 1$.

Proof uses construction of the Klain function (Hadwiger) and injectivity of the Klain embedding (Klain).

- Kl_Z is invariant under

$$\begin{pmatrix} 1 & * & * \\ & \ddots & * \\ & & 1 \end{pmatrix}.$$

Sketch of the proof

- Using the theory of highest weights Kl_Z is a line segment in the highest weight space.
- Let

$$\begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{pmatrix} \in \mathrm{SL}(V)$$

  act on Kl_Z to compute the highest weight of W and find $W = \wedge^k V$.
- To show $k \in \{0, 1, n-1, n\}$ use that the Klain embedding is a proper subset of $C(\mathrm{Gr}_k(n))$ if $1 < k < n-1$ (Alesker–Bernstein).

Omitting translation invariance: New examples

1. For $p \in \mathbb{N}$ define $M^p \colon \mathcal{K}(V) \to \mathcal{K}(\mathrm{Sym}^p V)$ by

$$h_{M^p K} \colon (\mathrm{Sym}^p(V))^* \to \mathbb{R}, \quad u \mapsto \int_K |\langle u, x^{\odot p} \rangle| dx.$$

   For $p = 1$ this is known as the moment body.

2. For $p, q \in \mathbb{N}$ define

$$G_{p,q} \colon \mathcal{K}_{(o)}(V) \to \mathcal{K}(\mathrm{Sym}^p V \otimes \mathrm{Sym}^q V^*)$$

   by

$$h_{G_{p,q} K}(u) = \int_{\mathrm{nc}(K)} |\langle u, x^{\odot p} \otimes y^{\odot q} \rangle| \langle x, y \rangle^{-q} i_x \mathrm{vol}.$$

Thank you!