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Ryll-Wojtaszczyk homogeneous polynomials on strictly convex circular domains in $\mathbb{C}^n$

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This is a joint work with Piotr Kot

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Outline

- History of Ryll-Wojtaszczyk homogeneous polynomials on the unit ball in $\mathbb{C}^n$
- Application of Ryll-Wojtaszczyk homogeneous polynomials in the construction of an inner function in the unit ball in $\mathbb{C}^n$
- Homogeneous polynomials on circular, strictly convex domains
- Applications of homogeneous polynomials on circular, strictly convex domains
  - Inner functions
  - Radon inversion problem for holomorphic functions
  - Holomorphic functions with divergent series of Taylor coefficients

Basic notation

- $\mathbb{B}_n = \{z \in \mathbb{C}^n \colon ||z|| < 1\}$ the unit ball in $\mathbb{C}^n$
- $\mathbb{S}_n = \{z \in \mathbb{C}^n \colon ||z|| = 1\}$ the unit sphere in $\mathbb{C}^n$
- σ - normed Lebesgue's measure on $\mathbb{S}_n$, i.e. $\sigma(\mathbb{S}_n) = 1$
- $\mathcal{O}(\mathbb{B}_d)$ space of functions holomorphic in $\mathbb{B}_n$
- $\mathcal{H}_p(\mathbb{B}_n) = \left\{ f \in \mathcal{O}(\mathbb{B}_n) \colon \quad \sup_{0<r<1} \left(\int_{\mathbb{S}_n} |f(rz)|^p d\sigma(z) \right)^{\frac{1}{p}} < \infty \right\}, \quad 0 < p < \infty$
- $\mathcal{H}_\infty(\mathbb{B}_n) = \left\{ f \in \mathcal{O}(\mathbb{B}_n) \colon \quad \sup_{z \in \mathbb{S}_n} |f(z)| < \infty \right\}$
- $||f||_p = \left(\int_{\mathbb{S}_n} |f(z)|^p d\sigma(z) \right)^{\frac{1}{p}}, \ 1 \le p < \infty, \quad ||f||_\infty = \sup_{z \in \mathbb{S}_n} |f(z)|.$

Basic notation cont.

Let $\Omega \subset \mathbb{C}^n$ be a domain with a defining function ρ of the class $\mathcal{C}^2$. We say that Ω is stictly convex in $P \in \partial\Omega$, if

$$2\mathrm{Re}\left(\sum_{j,k=1}^{n} \frac{\partial^2 \rho}{\partial z_j \partial z_k}(P) w_j w_k\right) + 2\left(\sum_{j,k=1}^{n} \frac{\partial^2 \rho}{\partial z_j \partial \overline{z}_k}(P) w_j \overline{w}_k\right) > 0$$

for any $w \in \mathbb{C}^n$ such that $\displaystyle\sum_{j=1}^{n} \frac{\partial \rho}{\partial z_j}(P) w_j = 0.$

We say that Ω is a circular domain, if $\lambda z \in \Omega$ for any $\lambda \in \partial\mathbb{D}$ and $z \in \Omega$.

Let $f \in \mathcal{O}(\Omega)$. For $z \in \partial\Omega$ function $f_z : \mathbb{D} \ni \lambda \longmapsto f(\lambda z)$ is called a slice function of f.

We say that p_m is a homogeneous polynomial of degree $m \in \mathbb{N}$, if

$$p_m(\lambda z) = \lambda^n p_m(z), \quad \lambda \in \mathbb{C},\ z \in \mathbb{C}^n.$$

Observe that if $\mathcal{O}(\Omega) \ni f(z) = \sum_{m=0}^{\infty} p_m(z)$, then $f_z(\lambda) = \sum_{m=0}^{\infty} p_m(z)\lambda^m$, i.e. $p_m(z)$ are Taylor coefficients of the function f_z.

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Ryll-Wojtaszczyk homogeneous polynomials

We say that polynomial p_m of degree $m \in \mathbb{N}$ is a ***Ryll-Wojtaszczyk polynomial*** (RW-polynomial), if

$$\left|\left|p_m\right|\right|_\infty = 1 \quad \text{and} \quad \left|\left|p_m\right|\right|_2 \geq 2^{-n}\sqrt{\pi}.$$

Existence of such polynomials for any $m \in \mathbb{N}$ was proved by Jerzy Ryll and Przemysław Wojtaszczyk in 1983 in the paper
***On homogeneous polynomials on a complex ball*, Trans. Amer. Math. Soc. 276 (1983), p. 107-116.**

Motivation: Is the identity map $id \colon \mathcal{H}_\infty(\mathbb{B}_n) \longmapsto \mathcal{H}_1(\mathbb{B}_n)$ a compact linear operator when $n > 1$? Answer: No.

There are at least 3 proofs of the existence of RW-polynomials

- Ryll, Wojtaszczyk, 1983 - 2 proofs
- Rudin, 1985

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Inner function in $\mathbb{C}^n$

Function $f \in \mathcal{O}(\mathbb{B}_n)$ is said to be inner, if

$$|f^*(z)| := \lim_{r \to 1^-} |f(rz)| = 1 \quad \sigma - a.e.\ on\ \mathbb{S}_n.$$

By use of RW-polynomials Aleksandrov was able to construct by induction a sequence of polynomials $\{R_k\}_{k \geq 0}$ with the following properties:

1. $R_k(0) = 0$
2. $\int_{\mathbb{S}_n} R_k \overline{R_l} d\sigma = 0$ for $k \neq l$
3. $|R_{k+1}| < 1 - \left| \sum_{j=0}^{k} R_j \right|$ on $\mathbb{S}_n$
4. $\int_{\mathbb{S}_n} |R_{k+1}|^2 d\sigma > 4^{-n} \int_{\mathbb{S}_n} \left(1 - \left| \sum_{j=0}^{k} R_j \right|\right)^2 d\sigma$

Theorem (Wojtaszczyk, 1997)

There exists $K = K(n) \in \mathbb{N}$ and a sequence of homogeneous polynomials p_m of degree m such that

1. $|p_n| \leq 2$ *on* $\mathbb{S}_n$
2. *for large* $s \in \mathbb{N}$ *we have* $\sum_{m=Ks}^{K(s+1)-1} |p_m| \geq \frac{1}{2}$ *on* $\mathbb{S}_n$.

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Homogeneous polynomials and inner functions on circular, strictly convex domains

From now Ω is a bounded, circular, strictly convex domain in $\mathbb{C}^n$.

Theorem (Kot, 2009)

There exists $K = K(\partial\Omega) \in \mathbb{N}$ such that there exists $N_0 \in \mathbb{N}$ such that for all integers $N \geq N_0$ and $n_1, \dots, n_K \in \mathbb{N}$ with $N \leq n_1 \leq \dots \leq n_K \leq 2N$ there exist homogeneous polynomials $u_1, \dots, u_K$ of degrees $n_1, \dots, n_K$, respectively, such that
$\frac{1}{2} < \max_{1 \leq j \leq K} |u_j(z)| < 1$ *for all $z \in \partial\Omega$.*

Theorem (Kot, 2017)

There exists an inner function $f \in \mathcal{O}(\Omega)$ such that for all $z \in \partial\Omega$ we have the following properties:

1. $|f_z^*| = 1$ *a.e. on $\partial\mathbb{D}$*
2. *f_z has a series of Taylor coefficients divergent with every power $s \in [0, 2)$, i.e. if $f_z(\lambda) = \sum_{n=1}^{\infty} p_n(z)\lambda^n$, then $\sum_{n=1}^{\infty} |p_n(z)|^s = \infty$ for $s < 2$.*

In above theorem 1 may be replaced with any strictly positive lower semi-continuous function on $\partial\Omega$.

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K-summing polynomials and lacunary K-summing polynomials

Let $K \in \mathbb{N}$. We say that $Q = \sum_{j=1}^{K} u_j$, where $\deg(u_j) = n_j \in \mathbb{N}, \quad j = 1, 2, \ldots, K$, is a K-summing polynomial, if it possesses the following properties:

i) $\max\limits_{1 \leq j \leq K} |u_j(z)| \leq 1 \quad \text{for} \quad z \in \partial\Omega$

ii) $\frac{1}{2}\deg(Q) \leq n_1 < n_2 < \cdots < n_K = \deg(Q)$.

We say that Q is a lacunary K-summing polynomial, if it is a K-summing polynomial and the following conditions hold:

iii) $\max\limits_{1 \leq j \leq K} |u_j(z)| \geq \dfrac{1}{2} \quad \text{for} \quad z \in \partial\Omega$

iv) $\sqrt[K]{2} < \dfrac{n_{j+1}}{n_j} < 2, \quad j = 1, 2, \ldots, K-1.$

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Operators $\mathcal{R}^p$ i $\mathcal{S}^p$

Let $\Omega \subset \mathbb{C}^n$ be a bounded, balanced, strictly convex domain with the boundary of the class $\mathcal{C}^2$. Fix $p > 0$. For a holomorphic function $f \in \mathcal{O}(\Omega)$ we may consider the integral operator $\mathcal{R}^p$ defined as follows

$$\mathcal{R}^p(f)(z) := \int_0^1 |f(zt)|^p dt, \quad z \in \partial\Omega$$

and called *Radon operator.* Then for a given strictly positive, continuous function $\Phi\colon \partial\Omega \longmapsto \mathbb{R}_+$ we look for a function $f \in \mathcal{O}(\Omega)$ such that

$$\mathcal{R}^p(f) = \Phi \quad \text{on} \quad \partial\Omega.$$

For a sequence $\{u_k\}_{k\in\mathbb{N}}$ of homogeneous polynomials of degree $n_k \in \mathbb{N}$ respectively, we may define the operator $\mathcal{S}^p$ as follows

$$\mathcal{S}^p\colon \ \mathcal{O}(\Omega) \ni \sum_{k=1}^{\infty} u_k \longmapsto \sum_{k=1}^{\infty} \frac{u_k}{\sqrt[p]{pn_k+1}} \in \mathcal{O}(\Omega).$$

Notice that if $f = \sum_{k=1}^{\infty} u_k$, then $\mathcal{S}^p(f)_z(\lambda) = \sum_{k=1}^{\infty} \frac{u_k(z)}{\sqrt[p]{pn_k+1}} \lambda^{n_k}$, so $\frac{u_k(z)}{\sqrt[p]{pn_k+1}}$ are Taylor coefficients of slice functions of $\mathcal{S}^p(f)$.

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Properties of lacunary $K-$summing polynomials

There exists $N \in \mathbb{N}$ large enough such that for any lacunary K-summing polynomial Q of degree greater than N, bounded continuous functions f, g and $\varepsilon \in (0,1)$ the following conditions hold on $\partial\Omega$

- $\left|\mathcal{R}^p\Big(\deg(Q)fQ+g\Big)-\mathcal{R}^p\Big(\deg(Q)fQ\Big)-\mathcal{R}^p(g)\right|<\varepsilon$
- $\left|\mathcal{R}^p\Big(\deg(Q)fQ\Big)-|f|^p\mathcal{R}^p(\deg(Q)Q)\right|<\varepsilon$
- $\left|\mathcal{S}^p\Big(\sqrt[p]{p\deg(Q)}fQ\Big)-f\mathcal{S}^p\Big(\sqrt[p]{p\deg(Q)}Q\Big)\right|<\varepsilon,\quad f$-polynomial
- $\left|\mathcal{S}^p\left(\sqrt[p]{p\deg(Q)}Q\right)\right|<K\sqrt[p]{2}$
- there exist constants c_K, C_K such that

  $$c_K \;\le\; \int_0^1 p\deg(Q)\Big|Q(zt)\Big|^p dt \;\le\; C_K,\quad z\in\partial\Omega$$

Radon inversion problem on circular, strictly convex domains

Theorem (Kot, P., 2022)

Let $p > 0$ and Φ be a strictly positive, continuous function on $\partial\Omega$. There exists a function $G \in \mathcal{O}(\Omega)$ such that $\mathcal{R}^p(G)(z) = \Phi(z)$ for $z \in \partial\Omega$ and $\mathcal{S}^p(G) \in \mathcal{C}(\overline{\Omega})$.

Sketch of the proof

1. There exist $\delta \in (0,1)$ and $\gamma > 0$ such that if φ is a strictly positive, continuous function on $\partial\Omega$ and ε, then there exists a polynomial $F := \sum_{m=1}^{N} c_m f_m Q_m$, where Q_m are lacunary $K-$summing polynomials, f_m are polynomials such that $\frac{\psi}{2N} < f_m < \frac{\psi}{N}$ on $\partial\Omega$ and c_m-constants, with the following properties:
   (c1) $\delta\Psi < \mathcal{R}^p(F) < \Psi$ on $\partial\Omega$
   (c2) $\left|\mathcal{R}^p(g+F) - \mathcal{R}^p(g) - \mathcal{R}^p(F)\right| < \varepsilon$ for any bounded and continuous g
   (c3) $\left|\mathcal{S}^p(F)\right| < \gamma(\Psi)^{\frac{1}{p}}$ on $\partial\Omega$.
2. Construct by induction a sequence of polynomials $\left\{F_j\right\}_{j\geq 0}$ such that on $\partial\Omega$
   - $\left(1-\frac{\delta}{2}\right)\left(\Phi - \mathcal{R}^p\left(\sum_{i=0}^{j-1} F_i\right)\right) > \Phi - \mathcal{R}^p\left(\sum_{i=0}^{j} F_i\right) > 0$
   - $\left|\mathcal{S}^p(F_j)\right| < \gamma\left(1-\frac{\delta}{2}\right)^{\frac{j}{p}} ||\Phi||_{\partial\Omega}^{\frac{1}{p}}$
3. $G := \sum_{j=0}^{\infty} F_j$ satisfies the Theorem.

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Radon inversion problem on circular, strictly convex domains

Theorem (Kot, P., 2022)

Let $p > 0$ and Φ be a strictly positive, continuous function on $\partial\Omega$. There exists a function $G \in \mathcal{O}(\Omega)$ such that $\mathcal{R}^p(G)(z) \ = \ \Phi(z)$ for $z \in \partial\Omega$ and $\mathcal{S}^p(G) \in \mathcal{C}(\overline{\Omega})$.

Sketch of the proof

1. There exist $\delta \in (0,1)$ and $\gamma > 0$ such that if φ is a strictly positive, continuous function on $\partial\Omega$ and ε, then there exists a polynomial $F := \sum_{m=1}^{N} c_m f_m Q_m$, where Q_m are lacunary $K-$summing polynomials, f_m are polynomials such that $\frac{\psi}{2N} < f_m < \frac{\psi}{N}$ on $\partial\Omega$ and c_m-constants, with the following properties:
   - (c1) $\delta\Psi < \mathcal{R}^p(F) \ < \ \Psi$ on $\partial\Omega$
   - (c2) $\Big|\mathcal{R}^p(g+F) - \mathcal{R}^p(g) - \mathcal{R}^p(F)\Big| < \varepsilon$ for any bounded and continuous g
   - (c3) $\Big|\mathcal{S}^p(F)\Big| < \gamma(\Psi)^{\frac{1}{p}}$ on $\partial\Omega$.
2. Construct by induction a sequence of polynomials $\left\{F_j\right\}_{j\geq 0}$ such that on $\partial\Omega$
   - $\left(1-\frac{\delta}{2}\right)\left(\Phi - \mathcal{R}^p\left(\sum_{i=0}^{j-1} F_i\right)\right) \ > \ \Phi - \mathcal{R}^p\left(\sum_{i=0}^{j} F_i\right) \ > \ 0$
   - $\Big|\mathcal{S}^p(F_j)\Big| < \gamma\left(1-\frac{\delta}{2}\right)^{\frac{j}{p}} ||\Phi||_{\partial\Omega}^{\frac{1}{p}}$
3. $G := \sum_{j=0}^{\infty} F_j$ satisfies the Theorem.

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Theorem (Kot, P., 2022)

Let Φ be a strictly positive continuous function on $\partial\Omega$. There exists a holomorphic function $f = \sum_{n=0}^{\infty} p_n$, where p_n are homogeneous polynomials, such that

1. $\mathcal{R}^p(f) = \Phi \quad on \quad \partial\Omega$
2. *every slice function of $\mathcal{S}^p(f)$ has a divergent series of Taylor coefficients with every exponent $s < \min\{1, p\}$, i.e.*
$$\sum_{n=0}^{\infty} \left(\frac{|p_n(z)|}{\sqrt[p]{pn+1}} \right)^s = \infty, \quad s < \min\{1, p\}, \; z \in \partial\Omega$$
3. *if $p \leq 1$, then $\mathcal{S}^p(f) \in \mathcal{C}(\overline{\Omega})$*
4. *if $p \in (1, 2]$, then $\mathcal{S}^p(f) \in L^2(z\partial\mathbb{D}), \; z \in \partial\Omega$.*

Due to orthogonality of p_n the condition 2 holds also for $p = 2$.
Conditions $1 - 2$ are satisfied also when Φ is lower semi-continuous.

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