Around Bezout inequalities for mixed volumes

Maud Szusterman,

Univ. Paris - Sorbonne Unversité

June 2022

Conference OLE - Geometry, Analysis and Convexity, Sevilla

Mixed volume : Minkowski's definition

Denote by $\mathcal{K}_n = \{K \subset \mathbb{R}^n : K \text{ compact convex set}\}$.

Let $K, L \in \mathcal{K}_n$. Then $Vol_n(\lambda K + \mu L)$ is a polynomial in (λ, μ) :

$$Vol_n(\lambda K + \mu L) = \sum_{k=0}^{n} \binom{n}{k} v_k \lambda^k \mu^{n-k}$$

where $v_k = V_n(K[k], L[n-k]) = V_n(K, ..., K, L, ..., L)$ are called mixed volumes.

Mixed volume : Minkowski's definition

- Let $K, L \in \mathcal{K}_n$. Then $Vol_n(\lambda K + \mu L) = \sum_{k=0}^{n} \binom{n}{k} v_k \lambda^k \mu^{n-k}$
- Let $K_1, ..., K_m \in \mathcal{K}_n$. Then :

$$Vol_n(\lambda_1 K_1 + \cdots + \lambda_m K_m) = \sum_{\substack{a=(a_1,\ldots,a_m)\\|a|=n}} \binom{n}{a} v_a \ \lambda^a$$

where $v_a = V_n(K_1[a_1], \ldots, K_m[a_m])$ are called mixed volumes.

Mixed volume : one or two properties

- Let $K, L \in \mathcal{K}_n$. Then $Vol_n(\lambda K + \mu L) = \sum_{k=0}^{n} \binom{n}{k} v_k \lambda^k \mu^{n-k}$
- Let $K_1, ..., K_m \in \mathcal{K}_n$. Then :

$$Vol_n(\lambda_1 K_1 + \cdots + \lambda_m K_m) = \sum_{\substack{a=(a_1,\ldots,a_m) \\ |a|=n}} \binom{n}{a} v_a \ \lambda^a$$

  where $v_a = V_n(K_1[a_1], \ldots, K_m[a_m])$ are called mixed volumes.
- $V_n : \mathcal{K}_n^n \to [0, +\infty)$ is a multilinear, continuous functional.

  Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be an affine transform. Then :

$$V_n(TK_1, ..., TK_n) = det(T) V_n(K_1, ..., K_n)$$

Bezout inequality

Let $f_1, ..., f_r : \mathbb{R}^n \to \mathbb{R}$ be polynomials. Denote by $X_1, ..., X_r$ the associated algebraic varieties $(X_i := \{x \in \mathbb{R}^n : f_i(x) = 0\})$.

The *Bezout inequality* states that :

$$deg(X_1 \cap ... \cap X_r) \leq \prod deg(X_i) \qquad [B]$$

Bezout inequality

Let $f_1, ..., f_r : \mathbb{R}^n \to \mathbb{R}$ be polynomials. Denote by $X_1, ..., X_r$ the associated algebraic varieties .

The *Bezout inequality* states that :

$$deg(X_1 \cap ... \cap X_r) \leq \prod deg(X_i) \qquad [B]$$

Denote by $P_1, ..., P_r$ the Newton polytopes of $f_1, ..., f_r$

Bezout inequality

Let $f_1, ..., f_r : \mathbb{R}^n \to \mathbb{R}$ be polynomials. Denote by $X_1, ..., X_r$ the associated algebraic varieties .

The *Bezout inequality* states that :

$$deg(X_1 \cap ... \cap X_r) \leq \prod deg(X_i) \qquad [B]$$

Denote by $P_1, ..., P_r$ the Newton polytopes of $f_1, ..., f_r$

We can reformulate [B] within the language of mixed volumes :

$$V(P_1, ..., P_r, \Delta[n-r])V(\Delta)^{r-1} \leq \prod_{i=1}^{r} V(P_i, \Delta[n-1])$$

thanks to a theorem by Bernstein, Kushnirenko and Khovanskii.

Bezout inequality (again)

Let $f_1, ..., f_n : \mathbb{R}^n \to \mathbb{R}$ be polynomials.
Let $X = X_2 \cap ... \cap X_n$ of dimension 1, and $Y = X_1$ (codim.1) .
Then Bezout inequality :

$$deg(X \cap Y) \leq deg(X)deg(Y) \qquad [B]$$

translates to

$$V_n(P_1, ..., P_n)V_n(\Delta) \leq V_n(P_2, ..., P_n, \Delta)V_n(P_1, \Delta[n-1]).$$

(recover previous inequality [B], by using [B] $r - 1$ times)

A direct geometric proof of [B] inequality

$$V_n(L_1, ..., L_n)V_n(\Delta) \leq V_n(L_2, ..., L_n, \Delta)V_n(L_1, \Delta[n-1]).$$

Since the inequality is invariant under replacing L_1 with $\lambda L_1 + x$, we may assume $L_1 \subset \Delta$, and $r(\Delta, L_1) = 1$, which implies $h_{L_1}(u_j) = h_\Delta(u_j)$ for all outer normals u_j, $j \leq n+1$, of Δ.

- In this case :

$$V(L_1, \Delta[n-1]) = \frac{1}{n}\sum_{j=1}^{n+1} h_{L_1}(u_j)Vol_{n-1}(K^{u_j}) = V_n(\Delta)$$

- therefore [B] follows from monotonicity of mixed volume.

More general Bezout inequality

- Let $K, L \in \mathcal{K}_n$. The inradius of K relative to L is $r(K, L) := \max\{\lambda > 0 : x + \lambda L \subset K, x \in \mathbb{R}^n\}$.
- A corollary of Diskant's inequality :

$$r(K, L)^{-1} \leq n\frac{V_1(K, L)}{Vol(K)} = n\frac{V(K[n-1], L)}{Vol(K)}$$

- Using this, J. Xiao has shown (2019) :

$$V(L_1, ..., L_n)V(K) \leq nV(L_2, ..., L_n, K)V(L_1, K[n-1])$$

for any convex bodies $L_1, ..., L_n$, and for any K.

More general Bezout inequality

- Let $K, L \in \mathcal{K}_n$. The inradius of K relative to L is $r(K, L) := \max\{\lambda > 0 : x + \lambda L \subset K, x \in \mathbb{R}^n\}$.
- A corollary of Diskant's inequality :

$$r(K, L)^{-1} \leq n \frac{V_1(K, L)}{Vol(K)} = n \frac{V(K[n-1], L)}{Vol(K)}$$

- Using this, J. Xiao has shown (2019) :

$$V(L_1, ..., L_n)V(K) \leq nV(L_2, ..., L_n, K)V(L_1, K[n-1])$$

for any convex bodies $L_1, ..., L_n$, and for any K.

- $K = [0, 1]^n$ shows that n is sharp.

Proof of Xiao's upper bound

- Let $K, L \in \mathcal{K}_n$. The inradius of K relative to L is $r(K, L) := \max\{\lambda > 0 : x + \lambda L \subset K, x \in \mathbb{R}^n\}$.
- Replace L_1 with $L' := r(K, L_1)L_1 + x \subset K$ (L' maximally contained).
- $r(K, L_1)V(L_1, ..., L_n) = V(L', L_2, ..., L_n) \leq V(K, L_2, ..., L_n)$ (monotonicity)
- therefore :

$$\begin{aligned} V(L_1, ..., L_n) \leq & \quad r(K, L_1)^{-1} \quad V(K, L_2, ..., L_n) \\ & \leq \frac{nV(K[n-1], L_1)}{V_n(K)} V(K, L_2, ..., L_n). \end{aligned}$$

Bezout constants

We define :

$$b_2(K) = \max_{L_1, L_2} \frac{V(L_1, L_2, K[n-2])V(K)}{V(L_1, K[n-1])V(L_2, K[n-1])} \quad \geq 1$$

And similarly

$$b(K) = \max_{L_1, \ldots, L_n} \frac{V(L_1, \ldots, L_n)V(K)}{V(L_2, \ldots, L_n, K)V(L_1, K[n-1])} \quad \geq 1$$

Bezout constants

We define :

$$b_2(K) = \max_{L_1,L_2} \frac{V(L_1, L_2, K[n-2])V(K)}{V(L_1, K[n-1])V(L_2, K[n-1])} \geq 1$$

And similarly

$$b(K) = \max_{L_1,\dots,L_n} \frac{V(L_1, \dots, L_n)V(K)}{V(L_2, \dots, L_n, K)V(L_1, K[n-1])} \geq 1$$

So that :

- $b_2(\Delta) = b(\Delta) = 1$ (by BKK theorem, or directly with MV)
- $\forall K, 1 \leq b_2(K) \leq b(K)$;
- by [Diskant, Xiao] : $\max_K b(K) \leq n$.
- $\forall K,\ b(TK) = b(K)$, for any (full-rank) affine T.

Who are the minimizers ?

Question [SZ '15]

For which bodies do we have $b_2(K) = 1$?

Question [SSZ '18]

For which bodies do we have $b(K) = 1$?

SZ '15 $\rightarrow$ [Soprunov, Zvavitch] (2015)
SSZ '18 $\rightarrow$ [Saroglou, Soprunov, Zvavitch] (2018)

Who are the minimizers ?

Question [SZ '15] For which K, do we have $b_2(K) = 1$?

Question [SSZ '18] For which K do we have $b(K) = 1$?

- **Theorem**[SSZ '18] If $b(K) = 1$, then $K = \Delta$.

Who are the minimizers ?

Question [SZ '15] For which K, do we have $b_2(K) = 1$?

Question [SSZ '18] For which K do we have $b(K) = 1$?

- **Theorem**[SSZ '18] If $b(K) = 1$, then $K = \Delta$.
- this doesn't close former question, since $b_2(K) \leq b(K)$.
- ... open whether $\exists K \in \mathcal{K}_n$ with $b_2(K) < b(K)$.

Who are the minimizers ?

Question [SZ '15] For which K, do we have $b_2(K) = 1$?

Who are the minimizers ?

Question [SZ '15] For which K, do we have $b_2(K) = 1$?

- **Theorem**[SSZ '18] Let P be an n-polytope.
  If $b_2(P) = 1$, then $P = \Delta$.

Who are the minimizers ?

Question [SZ '15] For which K, do we have $b_2(K) = 1$?

- **Theorem**[SSZ '18]
  If $b_2(P) = 1$, then $P = \Delta$.
- **Prop**[SZ '15] if $b_2(K) = 1$, then $K \neq A + B \quad$ (with $A \not\equiv B$)
  (K cannot be decomposable)

A definition (by Saroglou, Soprunov and Zvavitch)

- Dfn : K is called decomposable if
  $\exists A, B \in \mathcal{K}_n$, $A \not\equiv K$, such that $K = A + B$.

  (equivalently : $\exists A, B \in \mathcal{K}_n$, $A \not\equiv B$, such that $K = A + B$.)

A definition (by Saroglou, Soprunov and Zvavitch)

- Dfn : K is called decomposable if $\exists A, B \in \mathcal{K}_n$, $A \not\equiv K$, such that $K = A + B$.

- Dfn : K is called weakly decomposable if there exists $L \in \mathcal{K}_n$, $L \not\equiv K$, such that $S_{K+L} << S_K$.

A definition (by Saroglou, Soprunov and Zvavitch)

- Dfn : K is called decomposable if $\exists A, B \in \mathcal{K}_n$, $A \not\equiv K$, such that $K = A + B$.

- Dfn : K is called weakly decomposable if there exists $L \in \mathcal{K}_n$, $L \not\equiv K$, such that $S_{K+L} << S_K$.
- example : if $K = A + B$ is decomposable, then it is weakly decomposable (take $L = A$).

A definition (by Saroglou, Soprunov and Zvavitch)

- Dfn : K is called decomposable if $\exists A, B \in \mathcal{K}_n$, $A \not\equiv K$, such that $K = A + B$.

- Dfn : K is called weakly decomposable if there exists $L \in \mathcal{K}_n$, $L \not\equiv K$, such that $S_{K+L} << S_K$.
- example : if $K = A + B$ is decomposable, then it is weakly decomposable.
- example : if P is a polytope, $P \neq \Delta$, then P is weakly decomposable.

A definition (by Saroglou, Soprunov and Zvavitch)

- Dfn : K is called decomposable if $\exists A, B \in \mathcal{K}_n$, $A \not\equiv K$, such that $K = A + B$.

- Dfn : K is called weakly decomposable if there exists $L \in \mathcal{K}_n$, $L \not\equiv K$, such that $S_{K+L} << S_K$.
- example : if $K = A + B$ is decomposable, then it is weakly decomposable.
- example : if P is a polytope, $P \neq \Delta$, then P is weakly decomposable.
- example : if ∂K is somewhere locally smooth, then K is weakly decomposable. ($\rightarrow$ Wulff shape argument)

Who are the minimizers ?

Question [SZ '15] For which K, do we have $b_2(K) = 1$?

- **Thm**[SSZ '18] Let $P \in \mathbf{Poly}_n$.Then $b_2(P) = 1 \Rightarrow P = \Delta$.
- **Thm**['15, '18] if $b_2(K) = 1$, then K cannot be *weakly decomposable* ($\to K \notin \mathcal{W}_n$)

Who are the minimizers ?

Question [SZ '15] For which K, do we have $b_2(K) = 1$?

- **Thm**[SSZ '18] Let $P \in \mathbf{Poly}_n$.Then $b_2(P) = 1 \Rightarrow P = \Delta$.
- **Thm**['15, '18] if $b_2(K) = 1$, then K cannot be *weakly decomposable* ($\rightarrow K \notin \mathcal{W}_n$)

  $\rightarrow$ excludes bodies with (somewhere) smooth boundary.

Who are the minimizers ?

Question [SZ '15] For which K, do we have $b_2(K) = 1$?

- **Thm**[SSZ '18] Let $P \in \mathbf{Poly}_n$.Then $b_2(P) = 1 \Rightarrow P = \Delta$.
- **Thm**['15, '18] if $b_2(K) = 1$, then K cannot be *weakly decomposable* ($\rightarrow K \notin \mathcal{W}_n$)

$\longrightarrow$ recovers characterization among polytopes,
since $\mathbf{Poly}_n \cap \mathcal{W}_n = \mathbf{Poly}_n \setminus \{\Delta\}$.

Who are the minimizers ?

Question [SZ '15] For which K, do we have $b_2(K) = 1$?

- **Thm**[SSZ '18] Let $P \in \textbf{Poly}_n$.Then $b_2(P) = 1 \Rightarrow P = \Delta$.
- **Thm**['15, '18] if $b_2(K) = 1$, then K cannot be *weakly decomposable* ($\to K \notin \mathcal{W}_n$)

- ... some more *restrictions*, eg : at most finitely many facets.

Question [SSZ '18] For which K do we have $b(K) = 1$?

- **Theorem**[SSZ '18] If $b(K) = 1$, then $K = \Delta$.

Who are the minimizers ?

Question [SZ '15] For which K, do we have $b_2(K) = 1$?

- **Thm**[SSZ '18] Let $P \in \mathbf{Poly}_n$.Then $b_2(P) = 1 \Rightarrow P = \Delta$.
- **Thm**['15, '18] if $b_2(K) = 1$, then K cannot be *weakly decomposable* ($\rightarrow K \notin \mathcal{W}_n$)

- ... some more *restrictions*, eg : at most finitely many facets.

Question [SSZ '18] For which K do we have $b(K) = 1$?

- **Theorem**[SSZ '18] If $b(K) = 1$, then $K = \Delta$.
  $\rightarrow$ proof uses Wulff shape bodies, a pointwise Aleksandrov differentiation lemma, and builds on above *restrictions*.

Some other necessary condition

Let K be a convex body, denote $\Omega = supp(S_K) \subset \mathbb{S}^{n-1}$. Let $\Omega = \cup_{d=0}^{n-1} \Omega_d$, where $\Omega_d = \{u \in \Omega : K^u \text{ is } d\text{-dimensional}\}$.

- Theorem [S. 2022+]
  Assume $S_K(\Omega_{n-2}) > 0$. Then $b_2(K) > 1$.

Some other necessary condition

Let K be a convex body, denote $\Omega = supp(S_K) \subset \mathbb{S}^{n-1}$. Let $\Omega = \cup_{d=0}^{n-1} \Omega_d$, where $\Omega_d = \{u \in \Omega : K^u \text{ is } d\text{-dimensional}\}$.

- Theorem [S. 2022+]
  Assume $S_K(\Omega_{n-2}) > 0$. Then $b_2(K) > 1$.
- Corollary : in $\mathbb{R}^3$, the simplex is the only minimizer of $b_2(K)$.

Some other necessary condition

Let K be a convex body, denote $\Omega = supp(S_K) \subset \mathbb{S}^{n-1}$. Let $\Omega = \cup_{d=0}^{n-1} \Omega_d$, where $\Omega_d = \{u \in \Omega : K^u \text{ is } d\text{-dimensional}\}$.

- Theorem [S. 2022+]
  Assume $S_K(\Omega_{n-2}) > 0$. Then $b_2(K) > 1$.
- Corollary : in $\mathbb{R}^3$, the simplex is the only minimizer of $b_2(K)$.
- (this was already known, as a by-product in [SSZ18])

An isoperimetric condition

Let $L \in \mathcal{K}_n$ be a k-dimensional. Denote :

$$Iso(L) := \frac{1}{k}\frac{Vol_{k-1}(\partial L)}{Vol_k(L)} =: \frac{1}{k}\frac{|\partial L|}{|L|}$$

Thm[S. 2022] If $b_2(K) = 1$, then :

For any facet F of K : $Iso(F) \leq Iso(K)$.

An isoperimetric condition

Let $L \in \mathcal{K}_n$ be a k-dimensional. Denote :

$$Iso(L) := \frac{1}{k}\frac{Vol_{k-1}(\partial L)}{Vol_k(L)} =: \frac{1}{k}\frac{|\partial L|}{|L|}$$

Thm[S. 2022] If $b_2(K) = 1$, then :

For any facet F of K : $Iso(F) \leq Iso(K)$.

(that is to say : for all $F \in \mathcal{F}_{n-1}(K)$: $\frac{|\partial F|}{|F|} \leq \frac{n-1}{n}\frac{|\partial K|}{|K|}$.)

An isoperimetric condition

Let $L \in \mathcal{K}_n$ be a k-dimensional. Denote :

$$Iso(L) := \frac{1}{k}\frac{Vol_{k-1}(\partial L)}{Vol_k(L)} =: \frac{1}{k}\frac{|\partial L|}{|L|}$$

Thm[S. 2022] If $b_2(K) = 1$, then :

For any facet F of K : $Iso(F) \leq Iso(K)$.

$\rightarrow$ recovers the "at most finitely many facets" restriction.

An isoperimetric condition

$\rightarrow$ recovers the "at most finitely many facets" restriction.

Indeed, if K has infinitely many facets, then many will satisfy $Iso(F) > Iso(K)$.

By the isoperimetric inequality :

$$Iso(L) = \frac{1}{d}\frac{|\partial L|}{|L|} = \frac{1}{d}\frac{|\partial L|}{|L|^{\frac{d-1}{d}}}\frac{1}{|L|^{1/d}} \geq \frac{|B_2^d|^{1/d}}{|L|^{1/d}}$$

thus if (F_k) is a sequence of facets with $Vol_{n-1}(F_k) \rightarrow 0$, then $Iso(F_k) \rightarrow +\infty$.

An isoperimetric condition

Let $L \in \mathcal{K}_n$ be a k-dimensional. Denote :

$$Iso(L) := \frac{1}{k} \frac{Vol_{k-1}(\partial L)}{Vol_k(L)} =: \frac{1}{k} \frac{|\partial L|}{|L|}$$

Thm[S. 2022] If $b_2(K) = 1$, then, for any affine transform T :

$$\text{For any facet } F \text{ of } K : \; Iso(TF) \leq Iso(TK).$$

(since $b_2(K)$ is affine invariant, while $\max_F \frac{Iso(F)}{Iso(K)}$, is not)

An isoperimetric condition

Let $L \in \mathcal{K}_n$ be a k-dimensional. Denote :

$$Iso(L) := \frac{1}{k}\frac{Vol_{k-1}(\partial L)}{Vol_k(L)} =: \frac{1}{k}\frac{|\partial L|}{|L|}$$

Thm[S. 2022] If $b_2(K) = 1$, then, for any affine transform T :

For any facet F of K : $Iso(TF) \leq Iso(TK)$.

- **Example** : the unit cube. It satisfies $Iso(C_n) = 2$, and so does any of its facets. Thus the criteria only allows to conclude $b_2(C_n) > 1$, after using an affine transform T.

An isoperimetric condition

Let $L \in \mathcal{K}_n$ be a k-dimensional. Denote :

$$Iso(L) := \frac{1}{k} \frac{Vol_{k-1}(\partial L)}{Vol_k(L)} =: \frac{1}{k} \frac{|\partial L|}{|L|}$$

Thm[S. 2022] If $b_2(K) = 1$, then, for any affine transform T :

$$\text{For any facet } F \text{ of } K: \; Iso(TF) \leq Iso(TK).$$

- **Question** : if $P \neq \Delta$, does there always exist

$$\text{an affine transform } T \text{ s.t. } \max_F \frac{Iso(TF)}{Iso(TP)} > 1 \quad ?$$

... any questions ? (... or answers ?)

Thank you for your attention !!