Discretization of geometric inequalities via the lattice point enumerator

geOmetry, anaLysis & convExity
June 20 – 24, 2022

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Brunn-Minkowski inequality

Theorem

Given $K, L \in \mathcal{K}^n$ we have $\mathrm{vol}(K+L)^{1/n} \geq \mathrm{vol}(K)^{1/n} + \mathrm{vol}(L)^{1/n}$.

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Naturally, due to the homogeneity of vol of degree n, for any $\lambda, \mu \geq 0$

$$\mathrm{vol}(\lambda K + \mu L)^{1/n} \geq \lambda \mathrm{vol}(K)^{1/n} + \mu \mathrm{vol}(L)^{1/n}.$$

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As a consequence of the (weighed) arithmetic and geometric means inequality, we can obtain:

Corollary

Given $K, L \in \mathcal{K}^n$ we have $\mathrm{vol}((1 - \lambda)K + \lambda L) \geq \mathrm{vol}(K)^{1-\lambda} \mathrm{vol}(L)^{\lambda}$ for any $0 \leq \lambda \leq 1$

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Note: The inequality can be extended to arbitrary non-empty compact sets, and even to more general measurable families.

Functional counterpart:

Theorem: Prékopa-Leindler inequality

Let $0 \leq \lambda \leq 1$ and let $f, g, h : \mathbb{R}^n \to \mathbb{R}^+$ be Lebesgue integrable functions verifying

$$h((1-\lambda)x + \lambda y) \geq f(x)^{1-\lambda} g(y)^{\lambda}$$

for all $x, y \in \mathbb{R}^n$.

Related inequalities

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for all $x, y \in \mathbb{R}^n$.Then

$$\int_{\mathbb{R}^n} h(x)dx \geq \left(\int_{\mathbb{R}^n} f(x)dx\right)^{1-\lambda} \left(\int_{\mathbb{R}^n} g(x)dx\right)^\lambda .$$

Related inequalities

Generalization of Prékopa-Leindler:

Theorem: Borell-Brascamp-Lieb inequality

Let $0 \leq \lambda \leq 1$, let $-1/n \leq p \leq \infty$ and let $f, g, h : \mathbb{R}^n \to \mathbb{R}^+$ be Lebesgue integrable functions verifying

$$h((1-\lambda)x + \lambda y) \geq \mathcal{M}_p^\lambda(f(x), g(y))$$

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$$\int_{\mathbb{R}^n} h(x)dx \geq \mathcal{M}^\lambda_{\frac{p}{np+1}}\left(\int_{\mathbb{R}^n} f(x)dx, \int_{\mathbb{R}^n} g(x)dx\right).$$

Brunn-Minkowski generalizations

In general $h_{K+L} = h_K + h_L$.

Definition (Firey (1962))

Let $p \geq 1$ and $K, L \in \mathcal{K}^n$ containing the origin in their interior. Then the p-sum $K +_p L$ is the unique convex body such that

$$h_{K+_pL} = \left(h_K^p + h_L^p\right)^{1/p}.$$

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Definition (Lutwak, Yang, Zhang (2012))

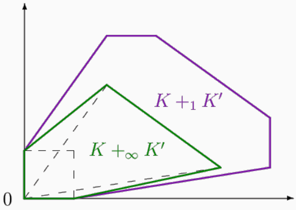
Let $K, L \subset \mathbb{R}^n$ be non-empty bounded sets and let $p \geq 1$. Then

$$K +_p L = \left\{(1-\mu)^{1/q}x + \mu^{1/q}y : x \in K, y \in L, \ \mu \in [0,1]\right\},$$

where $q \in [1, +\infty]$ is the Hölder conjugate of p, i.e., such that $1/p + 1/q = 1$.

Brunn-Minkowski generalizations

- $p = 1$: $K +_1 L = K + L$ (Minkowski addition).
- $p = \infty$: $K +_\infty L = \mathrm{conv}(K \cup L)$ (convex hull).
- If $p \leq q$ then:
  - $K +_q L \subset K +_p L$.
  - $(1 - \lambda) \cdot K +_p \lambda \cdot L \subset (1 - \lambda) \cdot K +_q \lambda \cdot L$.



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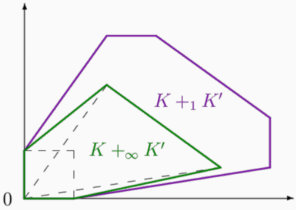

Theorem (Firey (1962), Lutwak, Yang, Zhang (2012))

Let $K, L \subset \mathbb{R}^n$ be non-empty bounded sets, and let $p \geq 1$. Then

$$\mathrm{vol}(K +_p L)^{p/n} \geq \mathrm{vol}(K)^{p/n} + \mathrm{vol}(L)^{p/n}.$$

Brunn-Minkowski generalizations

Definition

Let $f : \mathbb{S}^{n-1} \to \mathbb{R}^n_+$, the Wulff shape of f is

$$W(f) = \bigcap_{u \in \mathbb{S}^{n-1}} \{x \in \mathbb{R}^n : \langle x, u \rangle \leq f(u)\} .$$

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Then, for any $K \in \mathcal{K}^n$ containing the origin, $K = W(h_K)$. Thus, for any $K, L \in \mathcal{K}^n$ containing the origin and any $p \geq 1$,

$$(1-\lambda) \cdot K +_p \lambda \cdot L = W\left(((1-\lambda)h_K^p + \lambda h_L^p)^{1/p}\right).$$

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This definition can now be extended to $0 \leq p < 1$, in particular,

$$(1-\lambda) \cdot K +_0 \lambda \cdot L = W\big(h_K^{1-\lambda} h_L^{\lambda}\big).$$

Brunn-Minkowski generalizations

Böröczky, Lutwak, Yang and Zhang conjectured:

Conjecture - The log-Brunn-Minkowski inequality

Let $K, L \subset \mathbb{R}^n$ be centrally symmetric convex bodies, and let $\lambda \in (0, 1)$. Then

$$\mathrm{vol}\big((1-\lambda)\cdot K +_0 \lambda \cdot L\big) \geq \mathrm{vol}(K)^{1-\lambda}\mathrm{vol}(L)^{\lambda}. \tag{1}$$

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- $n = 2$ (Böröczky, Lutwak, Yang, Zhang, 2012)
- Unconditional bodies for $p = 0$ (Saroglou, 2015)
- Unconditional bodies for $0 < p < 1$ (Marsiglietti, 2015)
- Symmetric w.r.t. n independent hyperplanes (Böröczky, Kalantzopoulos, 2020)

Discretization preliminaries

Definition

A lattice $\mathcal{L}$ in $\mathbb{R}^n$ is a discrete additive subgroup of $\mathbb{R}^n$. Sometimes we will further require non-degeneracy (i.e. full dimensionality).

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The objects of study in this setting can be

- Discrete sets $A \subset \mathcal{L} \longrightarrow$ Cardinality $|A|$.
- Convex bodies $K \subset \mathcal{K}^n \longrightarrow$ Lattice point enumerator $G(K) = |K \cap \mathcal{L}|$.

Discreticing Brunn-Minkowski for the cardinality

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- Iglesias, Yepes Nicolás & Zvavitch (2020):

$$|A + B + \{0,1\}^n|^{1/n} \geq |A|^{1/n} + |B|^{1/n}.$$

Discreticing Brunn-Minkowski for $G(K)$

Theorem (Iglesias, Yepes Nicolás, Zvavitch (2020))

Let K, L be non-empty bounded sets and let $\lambda \in (0, 1)$. Then

$$G((1-\lambda)K + \lambda L + (-1,1)^n)^{1/n} \geq (1-\lambda)G(K)^{1/n} + \lambda G(L)^{1/n}.$$

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Taking $\lambda = 1/2$

Let K, L be non-empty bounded sets. Then

$G\left(\frac{K+L}{2} + [0, 1]^n\right) \geq \sqrt{G(K)G(L)}$ (Halikias, Klartag & Slomka)

$G\left(\frac{K+L}{2} + [0, 1]^n\right) \geq \frac{G(K)^{1/n} + G(L)^{1/n}}{2}$ (Iglesias, Yepes Nicolás & Zvavitch)

Linear results

Theorem (Iglesias, L., Yepes Nicolás (2020))

Let $t, s \geq 0$ and let $K, L \subset \mathbb{R}^n$ non-empty bounded sets such that $G(K)G(L) > 0$. Then

$$G(tK + sL + (-1, \lceil t + s \rceil)^n)^{1/n} \geq tG(K)^{1/n} + sG(L)^{1/n}. \qquad (2)$$

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that is, the classical Brunn-Minkowski inequality.

Theorem (Hernández Cifre, L., Yepes Nicolás (2021))

Let $\lambda \in (0,1)$ and $p \geq 1$, and let $K, L \subset \mathbb{R}^n$ be bounded sets with $G(K)G(L) > 0$. Then

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$$\mathrm{vol}((1-\lambda)\cdot K +_p \lambda \cdot L)^{p/n} \geq (1-\lambda)\mathrm{vol}(K)^{p/n} + \lambda \mathrm{vol}(L)^{p/n},$$

that is, the continuous L_p Brunn-Minkowski inequality.

L_0 results

Theorem (Hernández Cifre, L. (2021))

Let $K, L \subset \mathbb{R}^n$ be centrally symmetric convex bodies and let $\lambda \in (0,1)$. If either K, L are unconditional convex bodies or $n = 2$, then

$$G\left((1-\lambda)\cdot\left(K+\left[-\frac{1}{2},\frac{1}{2}\right]^n\right)+_0\lambda\cdot\left(L+\left[-\frac{1}{2},\frac{1}{2}\right]^n\right)+\left(-\frac{1}{2},\frac{1}{2}\right)^n\right)$$
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- The cubes cannot be reduced.
- It implies $\mathrm{vol}\big((1-\lambda)\cdot K +_0 \lambda\cdot L\big) \geq \mathrm{vol}(K)^{1-\lambda}\mathrm{vol}(L)^{\lambda}$, that is, the log-Brunn-Minkowski inequality, for both unconditional convex bodies or when $n = 2$.

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- It can be extended to $0 < p < 1$.

The isoperimetric inequality

Theorem

For every $K \in \mathcal{K}^n$ with non-empty interior we have

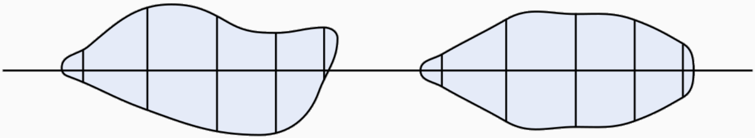
$$\frac{\mathcal{S}(K)^n}{\mathrm{vol}(K)^{n-1}} \geq \frac{\mathcal{S}(B_n)^n}{\mathrm{vol}(B_n)^{n-1}}.$$

Since $\mathcal{S}(B_n) = n\mathrm{vol}(B_n)$, then equivalently,

$$\mathcal{S}(K) \geq n\mathrm{vol}(K)^{1-\frac{1}{n}}\mathrm{vol}(B_n)^{\frac{1}{n}}.$$

The isoperimetric inequality

Classically, an argument of symmetrization (Steiner symmetrization) was used to prove the isoperimetric inequality.



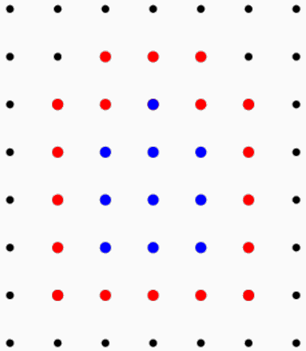
A similar technique (“compression”) is used in the discrete setting.

What is a discrete boundary?

We focus on the integer lattice, $\mathcal{L} = \mathbb{Z}^n$, from now on.

Definition

We define the boundary of a discrete set $A \subset \mathbb{Z}^n$ as $(A + \{-1, 0, 1\}^n) \setminus A$.



Discrete isoperimetric inequalities

The isoperimetric inequality admits the "neighbourhood form" given by

$$\mathrm{vol}(K + tB_n) \geq \mathrm{vol}(rB_n + tB_n),$$

where $r > 0$ is such that $\mathrm{vol}(rB_n) = \mathrm{vol}(K)$.

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We can generalize this by changing the “distance” involved (i.e. the symmetric convex body being summed):

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This allows us to extend the notion of isoperimetric inequalities to more general contexts.

Discrete isoperimetric inequalities

Theorem (Radcliffe, Veomett (2012))

Let $A \subset \mathbb{Z}^n$ be a non-empty finite set and let $r \in \mathbb{N}$ be such that $|\mathcal{I}_r| = |A|$. Then

$$|A + \{-1, 0, 1\}^n| \geq |\mathcal{I}_r + \{-1, 0, 1\}^n| .$$

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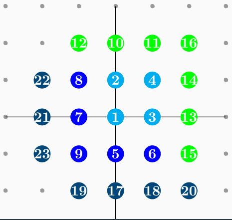
Definition

The extended lattice cube $\mathcal{I}_r$ is the set of the first r points of $\mathbb{Z}^n$ with respect to a suitably defined order.



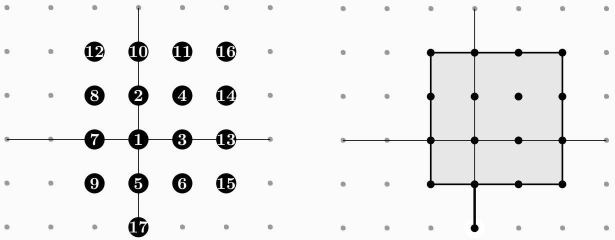
Discrete isoperimetric inequalities

In order to extend the result to convex bodies we define:

Definition

Given $r \in \mathbb{N}$, we denote

$$\mathcal{C}_r = \{(\lambda_1 x_1, \dots, \lambda_n x_n) \in \mathbb{R}^n : (x_1, \dots, x_n) \in \mathcal{I}_r, \lambda_i \in [0,1], i = 1, \dots, n\}.$$



The extended lattice cube $\mathcal{I}_{17}$ in $\mathbb{Z}^2$ (left) and the corresponding extended cube $\mathcal{C}_{17}$ in $\mathbb{R}^2$ (right).

Discrete isoperimetric inequalities

Theorem (Iglesias, L., Yepes Nicolás (2020))

Let $K \subset \mathbb{R}^n$ be a bounded set with $G(K) > 0$ and let $r \in \mathbb{N}$ be such that $G(\mathcal{C}_r) = G(K)$. Then

$$G\left(K + t[-1,1]^n\right) \geq G\left(\mathcal{C}_r + t[-1,1]^n\right) \tag{4}$$

for all $t \geq 0$.

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The discrete isoperimetric inequality (4) implies the classical isoperimetric inequality for non-empty compact sets.

Discretization of geometric inequalities via the lattice point enumerator

geOmetry, anaLysis & convExity
June 20 – 24, 2022

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