# Self-Similar Polygonal Tiling: extended abstract

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**Abstract**

A novel method for the construction of self-similar polygonal tilings, based on labeled rooted trees, will be discussed.

*Keywords:* tiling, self-similar, tree

## 1 Introduction

The goal of this abstract is to informally explain the discrete mathematics underlying tilings like those in Figures 1, 2 and 4. The individual polygonal tiles in each of these tilings are pairwise similar, and there are only finitely many

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up to congruence. Each tiling is *self-similar*. None of the tilings are periodic, yet each is *quasiperiodic*. These terms are defined below. Our method, based on rooted trees, is used to construct such tilings. It is a scheme that extends and simplifies previous tiling constructions.

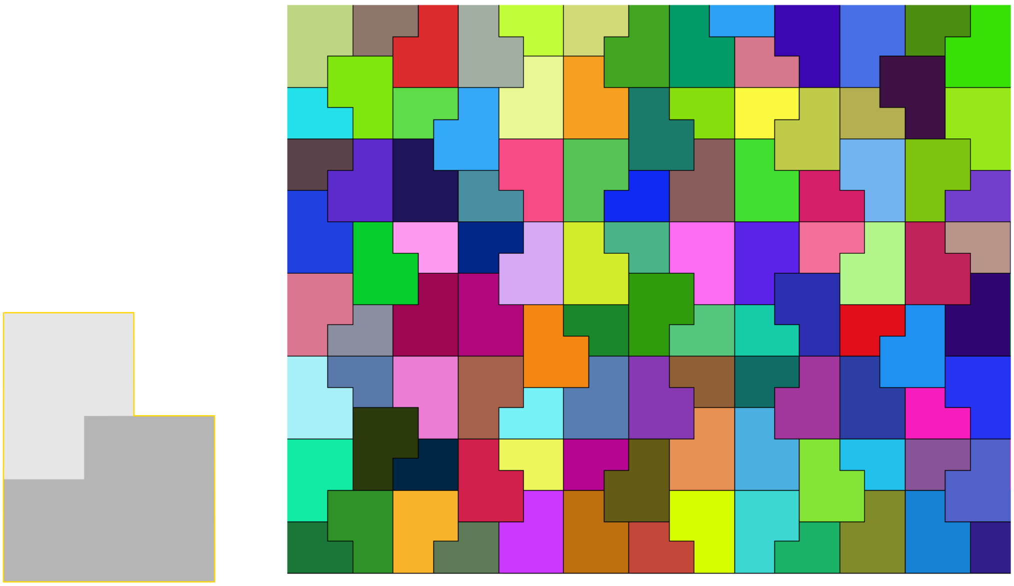

Fig. 1. A golden-b tiling.

On the left in Figure 1 are two similar tile shapes, shown in dark and light grey. The large (dark grey) tile (L), the small (light grey) tile (S), and their union, call it $G$, are pairwise similar polygons. The hexagon $G$, called the golden-b in [8], appears in [5], where it is attributed to Robert Ammann. If $\tau = (1+\sqrt{5})/2$ is the golden ratio, then the area of L is $\tau$ times larger than the area of S, and the area of $G$ is $\tau$ times larger than the area of L. With the exception of non-isosceles right triangles, the golden-b is the only polygon that can be partitioned into a non-congruent pair of scaled copies of itself [9]. The tiling on the right in the figure is a "patch" $P$ of the tiling $T$ of the entire plane by copies of the light and dark gray Ammann tiles. Congruent copies of this patch $Q$ appear "everywhere" in the tiling. More precisely, a patch congruent to $Q$ appears in every disk of sufficiently large radius, a property known as quasiperiodicity or repetitivity. Quasiperiodicity, less stringent than periodicity, has gained considerable attention since the 1984 Nobel Prize winning discovery of quasicrystals by Shechtman, Blech, Gratias, and Cahn [10]. The tiling $T$ in the figure is *self-similar* in that there exists a similarity transformation $\phi$ of the plane such that, for each tile $t \in T$, the "blown up" tile $\phi(t) = \{\phi(x) : x \in t\}$ is the non-overlapping union of the original tiles in $T$. There are uncountably many such golden-b tilings. Any patch of any one of these tilings, for example patch $Q$ in Figure 1, appears in every sufficiently large disk of every golden-b tiling, a property called *local isomorphism*. The ratio of large to small tiles in any ball of radius $R$ centered at a given arbitrary point tends to the golden ratio as $R \to \infty$.

The tilings in Figures 2 and 4 are also self-similar and quasiperiodic. In Figure 2, there are two similar tile shapes, the ratio of the sides of the larger quadrilateral to the smaller quadrilateral being $\sqrt{3} : 1$. In Figure 4 there are six similar tiles up to congruence.

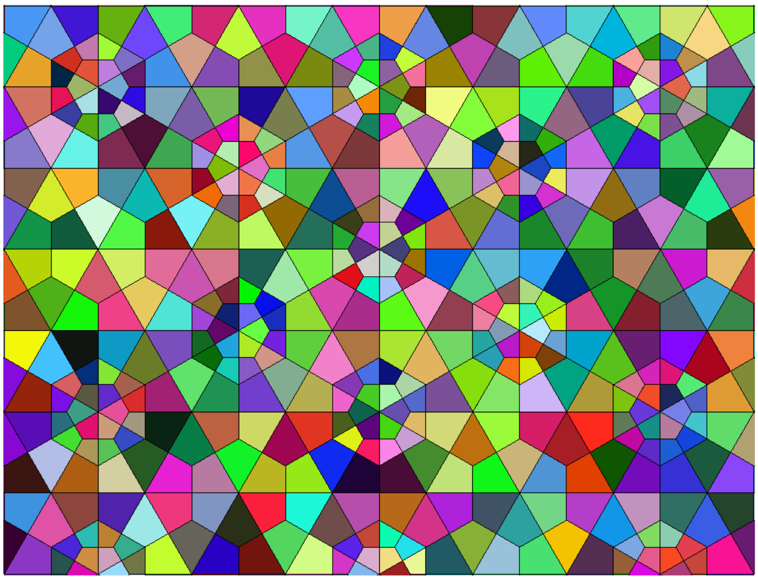

Fig. 2. A self-similar polygonal tiling by copies of two similar tiles.

## 2 Brief Background

There is a cornucopia of tilings of the plane possessing some sort of regularity. The mathematical literature is replete with papers on the subject, for example the tilings by regular polygons dating back at least to J. Kepler, tilings with large symmetry group as studied by Grünbaum and Shephard [5] and many others, and the aperiodic Penrose tilings and their relatives [7]. Selfsimilarity, in one form or another, has been intensely studied over the past few decades - arising in the areas of radix representation, symbolic dynamics, fractal geometry, Markov partitions, and wavelets; see for example [6,11,12].

For the purpose of this abstract, a **self-similar polygonal tiling** is a tiling $T$ of the plane by pairwise similar polygons such that

- there are finitely many tiles in $T$ up to congruence;
- $T$ is self-similar; and
- $T$ is quasiperiodic.

## 3 Methods

What follows is, up to some fine points, a short outline of our construction of self-similar tilings; for details see [1,2]. There are two relevant objects. First, let $P$ be a polygon satisfying certain properties detailed in the above cited papers and in the talk. Such a polygon will be referred to as a *generating polygon*. The golden-b in Figure 1, for example, is a generating polygon, and the rep-tiles, due to Golomb [3] and popularized by Gardner [4], are a special case.

Second, let $N \geq 2$ be an integer, $[N] = \{1, 2, \ldots, N\}$, and $[N]^\infty$ the set of all infinite sequences whose terms lie in $[N]$. Let $A = (a_1, a_2, a_3 \ldots) \in [N]^\infty$. We refer to the sequence $A$ as the *parameter*. In the construction of the tilings that follows, changing the parameter $A$ leads to a tiling in the same local isomorphism class.

For $a \in [N]$, define an *$a$-tree* as a rooted tree such that

(i) every non-leaf node except the root has $N$ children, and these children are labeled bijectively with $[N]$, and

(ii) the root has $N - 1$ children, and these children are labeled bijectively with $[N] \setminus \{a\}$.

Associated to a generating polygon $P$ and a parameter $A = (a_1, a_2, a_3, \ldots)$, there is a sequence $\mathcal{W} = (W_1, W_2, W_3, \ldots)$ such that $W_i$ is an $a_i$-tree for all $i$. Call such a sequence of trees an *$A$-sequence of trees*. With $N = 2$, Figure 3 shows the first four terms $W_1, W_2, W_3, W_4$ of an $A$-sequence of trees, where $A = (2, 1, 2, 1, \ldots)$.

Let $L$ be the set of all leaves of all the trees in an $A$-sequence $\mathcal{W}$, and let $l \in L$. The construction assigns to each triple $(P, A, l)$ a tile, and to each pair $(P, A)$ a tiling $T(P, A)$ of the plane. The main result is as follows.

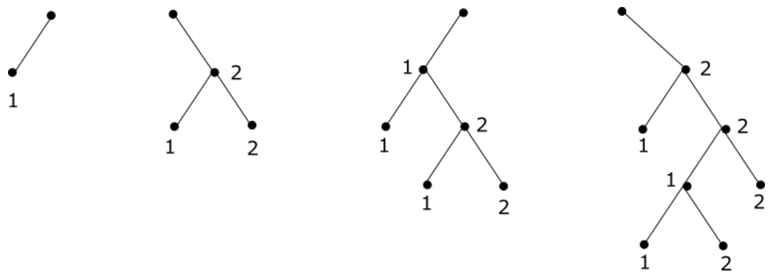


Fig. 3. The first four of an $A$-sequence of trees.

**Theorem** If $P$ is a generating polygon, then $T(P, A)$ is a self-similar tiling of the plane for infinitely may values of the parameter $A$.

The paper [2] provides precise results describing for exactly which parameters $A$ the tiling $T(P, A)$ is self-similar.

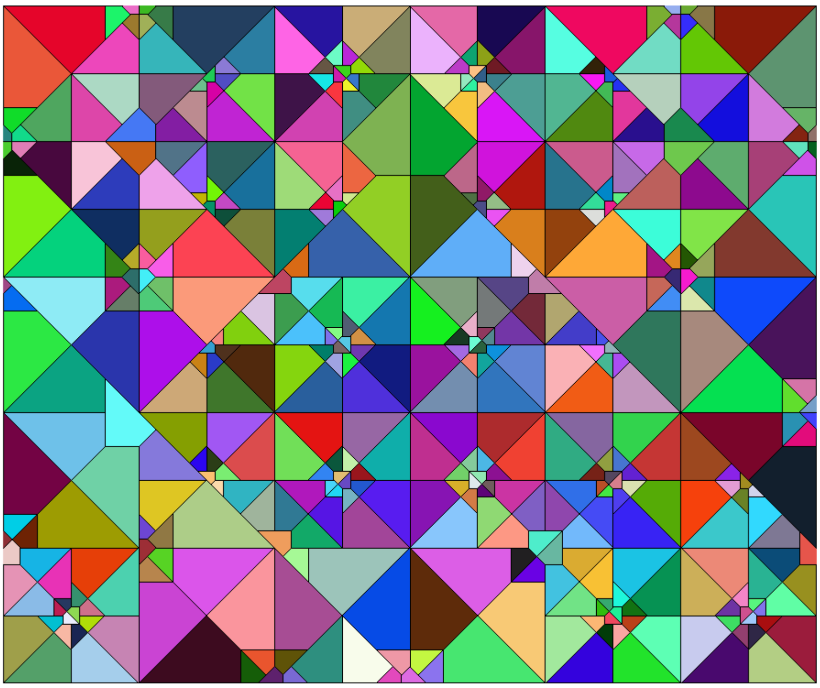

Fig. 4. A self-similar polygonal tiling by copies of 6 pairwise similar tiles.

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