Integral Free-Form Sudoku graphs

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Abstract

A free-form Sudoku puzzle is a square arrangement of $m \times m$ cells such that the cells are partitioned into m subsets (called blocks) of equal cardinality. The goal of the puzzle is to place integers $1, \ldots m$ in the cells such that the numbers in every row, column and block are distinct. Represent each cell by a vertex and add edges between two vertices exactly when the corresponding cells, according to the rules, must contain different numbers. This yields the associated free-form Sudoku graph. It was shown that all Sudoku graphs are integral graphs, in this paper we present many free-form Sudoku graphs that are still integral graphs.

Keywords: Sudoku, spectrum, eigenvectors.

1 Preliminaries

The $r-$regular slice $n-$sudoku puzzle is the free-form sudoku puzzle obtained from the $n-$sudoku puzzle by shifting the block cells in the $(in+d)^{th}$ row $(d-1)rn$ cells to the right, where $1 \leq d \leq n$. In Figure 1 (B), the cells are partitioned into 9 blocks denoted by B_i .
To study the eigenvalues of $r-$regular slice $n-$sudoku, let us start from the $n^2 \times n$ rectangular template slice where its cells partitioned into n^2 blocks and for $i = 0, 1, \ldots, n-1$, rows $in+1, in+2, \ldots (i+1)n$ contains only the n block numbers $in+1, in+2, \ldots (i+1)n$, with the additional restriction that the block numbers used in different $i-$collection of rows are distinct, see fig1 (A). Any $n \times m$ rectangular arrangement of cells that have been partitioned into r blocks can be represented by a graph whose vertices are the cells and where two vertices are adjacent if and only if the associated cells are in the same row, column or block of the arrangement, moreover one can view this graph as a layering of three graphs: the block layer (B) reflects the adjacencies due to the same block membership, the horizontal layer (H) reflect the adjacencies due to the same row membership and the vertical layer (V) reflect the adjacencies due to the same column membership.

The chosen layers into mutually exclusive cases immediately gives rise to a decomposition of the adjacency matrix of the slice template graph according to these three cases: $A = L_B + L_H + L_V$.
Starting from the slicing template and enlarge each block cell into $1 \times n$ cells belong to the same block we get slicing sudoku puzzle SSud(n,T), see Fig1 (A) and (B), a decomposition of the adjacency matrix of the slicing sudoku graph according to L_B, L_H and L_V is:

$$A^{\uparrow} = I_{n^3} \otimes (J_n - I_n) + L_B \otimes J_n + L_H \otimes J_n + L_V \otimes I_n,$$

where J_n is the all ones matrix.

Theorem 1.1 *The eigenvectors of $A^{\uparrow}$ can be partitioned into two types:*

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B1	B2	B3
B3	B1	B2
B2	B3	B1
B4	B5	B6
B6	B4	B5
B5	B6	B4
B7	B8	B9
B9	B7	B8
B8	B9	B7

(A) Template slice

B1	B1	B1	B2	B2	B2	B3	B3	B3
B3	B3	B3	B1	B1	B1	B2	B2	B2
B2	B2	B2	B3	B3	B3	B1	B1	B1
B4	B4	B4	B5	B5	B5	B6	B6	B6
B6	B6	B6	B4	B4	B4	B5	B5	B5
B5	B5	B5	B6	B6	B6	B4	B4	B4
B7	B7	B7	B8	B8	B8	B9	B9	B9
B9	B9	B9	B7	B7	B7	B8	B8	B8
B8	B8	B8	B9	B9	B9	B7	B7	B7

(B) 1-regular slice 3-sudoku

Fig. 1.

1 For any eigenvector x of L_V (corresponding to the eigenvalue λ) and $z \in ker(J_n)$, we have $x \otimes z$ is an eigenvector of $A^{\uparrow}$ corresponding to $\lambda - 1$, multiplicity of $\lambda - 1$ is equal to multiplicity of $\lambda(n-1)$.

2 For any eigenvector x of $nL_B+nL_H+L_V$ (corresponding to the eigenvalue α), the eigenvector $x \otimes 1_n$ is an eigenvectors of $A^{\uparrow}$ corresponding to $\alpha + n - 1$.

The proof is simple calculations

2 $r-$regular slice sudoku graphs are integral

We will denote by $A[i:j, r:k]$ the sub-matrix $[a_{ts}]$ with $i \le t \le j, r \le s \le k$ of A.

Theorem 2.1 *For slicing template corresponding to the $r-$regular slice $n-$sudoku puzzle*

$$L_H = I_{n^2} \otimes (J_n - I_n),$$
$$L_V = (J_{n^2} - I_{n^2}) \otimes I_n,$$
$$L_B = I_n \otimes \sum_{i=1}^{n-1} \pi^i \otimes \pi^{ir},$$

where $\pi = circ(0,1,0,\ldots,0)$, the circulant matrix associated to the vector $(0,1,0,\ldots,0)$.

Proof.

(i) For the principle submatrix of L_H, $L_H[in+1:(i+1)n, in+1:(i+1)n]$ represents the horizontal edges between the cells in the rows, $in+1:(i+1)n$ and columns $(1:n)$, so

$$L_H[in+1:(i+1)n, in+1:(i+1)n] = J_n - I_n, 0 \leq i \leq n-1,$$

The other entries in L_H are obviously zeros, so $L_H = I_{n^2} \otimes (J_n - I_n)$.

(ii) Same proof as above.

(iii) Since for each $i = 0, 1, \ldots, n-1$ the rows $[in+1:(i+1)n]$ in the slicing template contains n blocks, each one contains n cells, the rows contain all cells of its blocks, so in L_B graph whenever $i \neq j$, there is no edges between cells in the rows $[in+1:(i+1)n]$ and the cells in the rows $[jn+1:(j+1)n]$, so, L_B is diagonal of block matrices all of same size $(n^2 \times n^2)$.

Since for each $i = 0, 1, \ldots, n-1$ we have the same shifting in the rows $[in+1:(i+1)n]$, all matrices along the diagonal are identical. Therefore, $L_B = I_n \otimes B$ for a suitable matrix B.

In the matrix B, for fixed i the submatrix

$$B[in+1:(i+1)n, (i+1)n+1:(i+2)n]$$

describes the block rotation between row in and the next row $(i+1)n$. The rotation is r cells to the right, therefore

$B[in+1:(i+1)n, (i+1)n+1:(i+2)n] = \pi^r$.

In the same way, we consider the submatrix

$$B[in+1:(i+1)n, (i+j)n+1:(i+j+1)n+1].$$

Here we find that the block rotation is jr cells to the right, so

$$B[in+1:(i+1)n, (i+j)n+1:(i+j+1)n+1] = \pi^{rj}.$$

It follows that,

$$B = \textstyle\sum_{i=1}^{n-1} \pi^i \otimes \pi^{ri}$$

therefore,

$$L_B = I_n \otimes \sum_{i=1}^{n-1} \pi^i \otimes \pi^{ri}.$$

□

Let n be a fixed integer and set $\omega = e^{\frac{2\pi i}{n}}$. By the Fourier matrix of order n we shall mean the matrix

$$F_n = \frac{1}{\sqrt{n}}[f_{ij}], where\ f_{ij} = \overline{\omega^{(i-1)(j-1)}}.$$

It is well known [5] that

$$diag(1, \omega, \omega^2, \ldots, \omega^{n-1}) = F_n^* \pi F_n$$

and

$$diag(n, 0, 0, \ldots, 0) = F_n^* J_n F_n.$$

Theorem 2.2 *$r-$regular slice sudoku graph are integral graph.*

Proof. The eigenvalues of $r-$regular slice sudoku graphs can be totally determined by eigenvalues of L_V and $(nL_B + nL_H) + L_V$. The eigenvalues of L_V are integers since it is an adjacency matrix of n disconnected complete multipartite graphs, $K_{n,n,\ldots,n}$. Using Theorem1.1, the eigenvalues of $r-$regular slice sudoku $(n^2 \times n^2)$ Type 1 are integers.

For the second type of eigenvalues, suppose that

$$\phi = nL_B + nL_H + L_V.$$

Then,

$$\begin{aligned}
\phi &= (J_{n^2} - I_{n^2}) \otimes I_n + nI_{n^2} \otimes (J_n - I_n) + \left(nI_n \otimes \sum_{j=1}^{n-1} \pi^j \otimes \pi^{jr} \right) \\
&= J_{n^2} \otimes I_n + nI_{n^2} \otimes J_n - (n+1)I_{n^3} + \left(nI_n \otimes \sum_{j=1}^{n-1} \pi^j \otimes \pi^{jr} \right),
\end{aligned}$$

so

$$\begin{aligned}
&(F_n^* \otimes F_n^* \otimes F_n^*)\,\phi\,(F_n \otimes F_n \otimes F_n) = ((F_n^* J_n F_n) \otimes (F_n^* J_n F_n) \otimes I_n) + (nI_{n^2} \otimes (F_n^* J_n F_n)) \\
&- (n+1)I_{n^3} + \left(nI_n \otimes \sum_{j=1}^{n-1} \underbrace{F_n^* \pi^j F_n}_{(F_n^* \pi F_n)^j} \otimes \underbrace{F_n^* \pi^{jr} F_n}_{(F_n^* \pi F_n)^{jr}} \right) \\
&= (diag(n, 0, 0, \ldots, 0) \otimes diag(n, 0, 0, \ldots, 0) \otimes I_n) + (nI_{n^2} \otimes diag(n, 0, 0, \ldots, 0)) \\
&- (n+1)I_{n^3} + \left(nI_n \otimes \sum_{j=1}^{n} diag(1, \omega^j, \omega^{2j}, \ldots, \omega^{(n-1)j}) \otimes diag(1, \omega^{jr}, \omega^{2jr}, \ldots, \omega^{(n-1)jr}) \right).
\end{aligned}$$

Note that

$$diag(1, \omega^j, \omega^{2j}, \dots, \omega^{(n-1)j}) \otimes diag(1, \omega^{jr}, \omega^{2jr}, \dots, \omega^{(n-1)jr})$$
$$= diag(1, \omega^{jr}, \omega^{2jr}, \dots, \omega^{(n-1)jr}, \omega^j, \omega^{jr+j}, \omega^{2jr+j}, \dots, \omega^{j+(n-1)jr}, \dots).$$

Since

$$\sum_{j=1}^{n-1} diag(1, \omega^{jr}, \omega^{2jr}, \dots, \omega^{(n-1)jr}, \omega^j, \omega^{jr+j}, \omega^{2jr+j}, \dots, 1, \dots, \omega^{j+(n-1)jr}, \dots)$$
$$= diag(n-1, \sum_{j=1}^{n-1} \omega^{jr}, \dots, \sum_{j=1}^{n-1} \omega^{(n-1)jr}, \dots, \sum_{j=1}^{n-1} \omega^{j+(n-1)jr}, \dots)$$

and

$$\sum_{j=0}^{n-1} \omega^{kj} = \begin{cases} n-1 & k = 0, n, 2n, \dots \\ -1 & otherwise \end{cases}$$

we see that, $(F_n^* \otimes F_n^* \otimes F_n^*)\, \phi\, (F_n^* \otimes F_n^* \otimes F_n^*)$ is an integer diagonal matrix.$\square$

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