Two-line graphs of partial Latin rectangles

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Abstract

Two-line graphs of a given partial Latin rectangle are introduced as vertex-and-edge-coloured bipartite graphs that give rise to new autotopism invariants. They reduce the complexity of any currently known method for computing autotopism groups of partial Latin rectangles.

Keywords: partial Latin rectangle, autotopism group, entry invariant, adequate partition.

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1 Introduction

Let $[n] := \{1, \ldots, n\}$. An $r \times s$ *partial Latin rectangle* based on $[n]$ is an $r \times s$ array $L = (L[i, j])$ containing symbols from the set $[n] \cup \{\cdot\}$ such that each row and each column contains at most one copy of any symbol in $[n]$. The set of such rectangles is denoted $\mathrm{PLR}(r, s, n)$. The partial Latin rectangle L is uniquely determined by its *entry set*

$$\mathrm{Ent}(L) := \{(i, j, L[i, j]) : i \in [r], j \in [s], L[i, j] \in [n]\}.$$

Any triple $(i, j, L[i, j]) \in \mathrm{Ent}(L)$ is called an *entry* of L, and any pair $(i, j) \in [r] \times [s]$ is a *cell*. Any cell (i, j) in L containing the symbol $\cdot$ is said to be *empty*. If L does not have empty cells, then it is a *Latin rectangle*. This is a *Latin square of order* n if $r = s = n$.

For each $t \in \mathbb{N}$, let S_t denote the symmetric group on $[t]$. An *isotopism* $\theta = (\alpha, \beta, \gamma) \in S_r \times S_s \times S_n$ acts on the set $\mathrm{PLR}(r, s, n)$, by permuting the rows, columns, and symbols of any $L \in \mathrm{PLR}(r, s, n)$ by α, β, and γ, respectively. This is an *autotopism* of L if $\theta(L) = L$. The set $\mathrm{Atop}(L)$ of autotopisms of L forms a group, called its *autotopism group*. Although autotopism groups of Latin rectangles have received much attention in the literature (see [1,4,11,12,13] and many others), those of partial Latin rectangles have only recently been studied [5,7,9]. Methods for obtaining such groups are based either on a backtracking computation of non-polynomial time complexity, or on the computation of the automorphism group of a related graph [6,12].

The problem of computing the autotopism group of a (partial) Latin rectangle is, therefore, as difficult as solving the graph isomorphism problem [2,3]. In order to reduce as low as possible the related complexity, different autotopism invariants of (partial) Latin rectangles have been described [6,8,10,11,14]. All of them yield a series of partitions of the entries, rows, columns and/or symbols of the corresponding (partial) Latin rectangle so that all their parts are preserved by autotopisms. Nevertheless, no invariant whose corresponding partitions are optimal is currently known. Here we introduce new autotopism invariants that give rise to partitions that are closer to optimal than before.

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2 Adequate partitions and derived refinement

Let S be a finite set. From here on, let $a \sim_{\mathcal{P}} b$ denote the fact that two elements $a, b \in S$ are in the same part of a partition $\mathcal{P}$ of S.

Let $L = (L[i,j]) \in \mathrm{PLR}(r,s,n)$. Any partition $\mathcal{P}$ of the set $[r]$ constitutes a *row-partition* of L. This is *adequate* if it is preserved by any autotopism $(\alpha, \beta, \gamma) \in \mathrm{Atop}(L)$, that is, if $i \sim_{\mathcal{P}} \alpha(i)$, for all $i \in [r]$. The concepts of *(adequate) column-*, *symbol-* and *entry–partitions* are analogously introduced with respect to the sets $[s]$, $[n]$ and $\mathrm{Ent}(L)$, respectively. A *system of partitions* on L is a triple $\mathfrak{P} = (\mathcal{P}_{\mathrm{row}}, \mathcal{P}_{\mathrm{col}}, \mathcal{P}_{\mathrm{sym}})$, whose components are, respectively, a row-, a column- and a symbol-partition of L. This is *adequate* if its three components are. Further, $\mathfrak{P}$ produces a partition $E(\mathfrak{P})$ of the entry set $E(L)$ so that $(i,j,L[i,j]) \sim_{E(\mathfrak{P})} (i',j',L[i',j'])$ if and only if $i \sim_{\mathcal{P}_{\mathrm{row}}} i'$, $j \sim_{\mathcal{P}_{\mathrm{col}}} j'$ and $L[i,j] \sim_{\mathcal{P}_{\mathrm{sym}}} L[i',j']$. If we label the entries of L so that two entries have the same label whenever they belong to the same part of $E(\mathfrak{P})$, then we can define a new system of partitions $D(\mathfrak{P}) = (D(\mathcal{P}_{\mathrm{row}}), D(\mathcal{P}_{\mathrm{col}}), D(\mathcal{P}_{\mathrm{sym}}))$ of L. Here, two rows are in the same part of $D(\mathcal{P}_{\mathrm{row}})$ if and only if the multisets of labels corresponding to their entries coincide. The partitions $D(\mathcal{P}_{\mathrm{col}})$ and $D(\mathcal{P}_{\mathrm{sym}})$ are similarly defined. We call $D(\mathfrak{P})$ the *derivation* of $\mathfrak{P}$.

Example 2.1 The *row type*, *column type* and *symbol type* of $L \in \mathrm{PLR}(r,s,n)$ are defined [10,14], respectively, as the number of entries per row, the number of entries per column, and the number of appearances of each symbol in the set $[n]$ within L. The respective distributions of rows, columns and symbols of L according to these types determine a system of partitions on L, which we denote $\mathfrak{P}_T(L)$. Then, $D(\mathfrak{P}_T(L)) = \mathfrak{P}_{SEI}(L)$, where $\mathfrak{P}_{SEI}(L)$ is the system of partitions on L that arises from the so-called *strong entry invariants* [6,8]. $\lhd$

A system $\mathfrak{P}'$ defined on L is *finer* than $\mathfrak{P}$ if each component of $\mathfrak{P}$ is finer than the corresponding component of $\mathfrak{P}'$. We denote this fact as $\mathfrak{P}' \leq \mathfrak{P}$.

Theorem 2.2 *Let $\mathfrak{P}$ be a system of partitions on a partial Latin rectangle. Then, (i) if $\mathfrak{P}$ is adequate, so is $D(\mathfrak{P})$, (ii) $D(\mathfrak{P}) \leq \mathfrak{P}$, and (iii) if $\mathfrak{P}'$ is another system of partitions so that $\mathfrak{P}' \leq \mathfrak{P}$, then $D(\mathfrak{P}') \leq D(\mathfrak{P})$.*

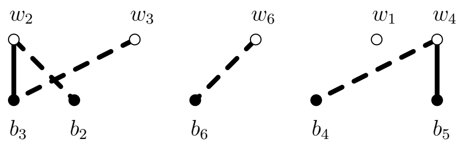
For each given adequate system of partitions $\mathfrak{P}$, let us consider the sequence $\ldots \leq \mathfrak{P}^{(2)} \leq \mathfrak{P}^{(1)} \leq \mathfrak{P}^{(0)} = \mathfrak{P}$, where $\mathfrak{P}^{(i)} := D\left(\mathfrak{P}^{(i-1)}\right)$, for each $i > 0$, until we find a positive integer $m > 0$ so that $\mathfrak{P}^{(m)} = \mathfrak{P}^{(m-1)}$. We call the resulting adequate system of partitions $D[\mathfrak{P}]$ the *derived refinement* of $\mathfrak{P}$.

3 The two-lines invariant

In this section we define a system of partitions whose derived refinement is finer than the derived refinement of the system of partitions arising from the strong entry invariants, which is described in Example 2.1. To this end, let $L = (L[i,j]) \in \mathrm{PLR}(r,s,n)$. For each pair (r_1, r_2) of different row indexes in L, we define their *two-row graph* of L as the vertex-and-edge-coloured bipartite graph $G_{r_1r_2}(L) := (V_{r_1}(L) \cup V_{r_2}(L), E_{r_1r_2}(L))$ as follows:

- For each $i \in \{r_1, r_2\}$, the set $V_i(L)$ is identified with the set $\{(i,j,L[i,j]) \in E(L) : j \in [s]\}$. The vertices in $V_{r_1}(L)$ are coloured white and labeled as $w_{L[r_1,j]}$, whereas those in $V_{r_2}(L)$ are colored black and labeled as $b_{L[r_2,j]}$.
- For each $j \in [s]$ such that $w_{L[r_1,j]} \in V_{r_1}(L)$ and $b_{L[r_2,j]} \in V_{r_2}(L)$, we draw the edge $w_{L[r_1,j]}b_{L[r_2,j]}$ as a solid line.
- For each $k \in [n]$ such that $w_k \in V_{r_1}(L)$ and $b_k \in V_{r_2}(L)$, we draw the edge w_kb_k as a dashed line.
- There are no other edges in the graph.

Example 3.1 Let $L \equiv \begin{array}{|c|c|c|c|c|c|c|c|}\hline 2 & \cdot & 3 & \cdot & 6 & \cdot & 1 & 4 \\ \hline 3 & 2 & \cdot & 6 & \cdot & 4 & \cdot & 5 \\ \hline \end{array} \in \mathrm{PLR}(2,8,6)$. Then,

$G_{12}(L) \equiv$ 

$\triangleleft$

Two-column and *two-symbol graphs* of L are similarly defined by taking the corresponding two-rows graphs of $L^{(12)}$ and $L^{(13)}$, respectively. Here, for each pair of distinct elements $i, j \in \{1,2,3\}$, the partial Latin rectangle $L^{(ij)}$ is obtained from L by swapping the ith and jth components of each of its entries. We use the general term *two-lines graph* for these three types of graphs. Since all of them consist of disjoint paths and cycles, verifying whether two graphs are isomorphic can be done in linear time complexity. Let us use this fact to construct new adequate systems of partitions.

If r_1, r_2, r_3, r_4 are rows in L such that $r_1 \neq r_2$ and $r_3 \neq r_4$, then we write $(r_1, r_2) \sim (r_3, r_4)$ if $G_{r_1 r_2}$ is isomorphic to $G_{r_3 r_4}$. Now, let $\mathcal{P}$ be an adequate row-partition of L. Then, we define an equivalence relation $\sim_{G(\mathcal{P})}$ on the rows of L as follows: $r_1 \sim_{G(\mathcal{P})} r_2$ if and only if

(i) $r_1 \sim_{\mathcal{P}} r_2$,

(ii) for any row $r_i \neq r_1$, there exists a row $r_j \neq r_2$ such that $r_i \sim_{\mathcal{P}} r_j$, and $(r_1, r_i) \sim (r_2, r_j)$, and

(iii) for any row $r_i \neq r_2$, there exists a row $r_j \neq r_1$ such that $r_i \sim_{\mathcal{P}} r_j$, and $(r_2, r_i) \sim (r_1, r_j)$.

This relation gives rise to an adequate row-partition $G(\mathcal{P}) \leq \mathcal{P}$, which can be refined again by repeating this procedure. In this way, we define a sequence $\ldots \leq \mathcal{P}_{(2)} \leq \mathcal{P}_{(1)} \leq \mathcal{P}_{(0)} = \mathcal{P}$, where $\mathcal{P}_{(i)} = G(\mathcal{P}_{(i-1)})$, for all $i > 0$, until no further refinement is achieved. We call the resulting adequate row-partition $G[\mathcal{P}]$ the *two-row graph refinement* of $\mathcal{P}$. Refinements of columns and symbols are described in a similar manner. Thus, from an adequate system of partitions $\mathfrak{P} = (\mathcal{P}_{\text{row}}, \mathcal{P}_{\text{col}}, \mathcal{P}_{\text{sym}})$, we compute the *two-line graph refinement* $G[\mathfrak{P}] := (G[\mathcal{P}_{\text{row}}], G[\mathcal{P}_{\text{col}}], G[\mathcal{P}_{\text{sym}}])$. This works even if $\mathfrak{P}$ is the system of singletons.

Theorem 3.2 *Let $\mathfrak{P}$ and $\mathfrak{P}'$ be two systems of partitions on a partial Latin rectangle L. If $\mathfrak{P}' \leq \mathfrak{P}$, then $G[\mathfrak{P}'] \leq G[\mathfrak{P}]$.*

Theorem 3.3 *The complexity of the two-line graph refinement of an adequate system of partitions on a partial Latin rectangle $L \in \mathrm{PLR}(r, s, n)$ in the worst case scenario is at most $O(M^6 \log M)$, where $M = \max(r, s, n)$.*

Theorem 3.4 *Let $\mathfrak{P}$ be an adequate system of partitions on a given partial Latin rectangle L. Then, $D[G[\mathfrak{P}]] \leq D[\mathfrak{P}_{SEI}(L)]$.*

The two-line graph refinement is particularly more effective than the strong entry invariants when dealing with very sparse or very dense partial Latin rectangles, since in those cases the strong entry invariants yield a very coarse system of partitions (or just the singleton system of partitions in the case of a Latin rectangle). Experimental evidence, obtained from running the algorithm on thousands of randomly generated partial Latin rectangles, shows that the two-line graph refinement yields systems of partitions that are close to optimal, and, in fact, optimal in most cases. This improves the current computational complexity of finding autotopism groups of random partial Latin rectangles.

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