Geometric quadrangulations of a polygon

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Abstract

We deal with a *geometric quadrangulation* of a polygon P. We define a new notion called "the spirality" of P, which measures how close P is from being a convex polygon. Using the spirality, we describe (1) a condition of P to admit a geometric quadrangulation, and (2) a condition of P guaranteeing that any two geometric quadrangulations on P can be related by a sequence of edge flips.

Keywords: Geometric graph, quadrangulation, polygon, edge flip

1 Introduction

Consider a *planar n-point set* S (i.e. a point set on the plane with $|S| = n$), which is always assumed to be in *general position* (i.e., no three points are collinear). Let $\mathrm{Conv}(S)$ denote the boundary of the convex hull of S. A *geometric triangulation* on S means a geometric plane graph whose vertex set is S, whose outer cycle coincides with $\mathrm{Conv}(S)$, and each of whose finite faces is triangular. (See the left of Figure 1.)

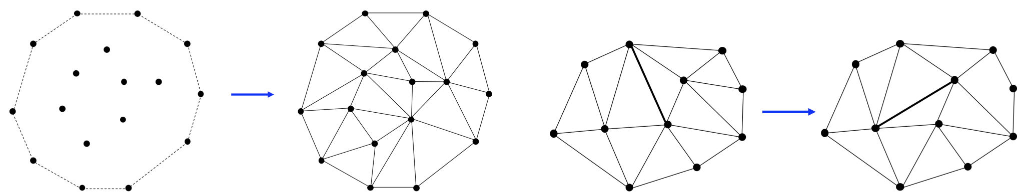

Fig. 1. Geometric triangulation on a point set S and an edge flip

It is easy to see that the following holds.

Proposition 1.1 *Every planar n-point set with $n \geq 3$ admits a geometric triangulation.*

An edge e of a triangulation is flippable if it belongs to the boundary of two triangular faces whose union is a convex quadrilateral C. *Flipping* e means substituting e by the second diagonal of C, see the right of Figure 1. Lawson proved the following theorem [7].

Theorem 1.2 (Lawson [7]) *Given a planar point set S with $|S| \geq 3$, any two geometric triangulations on S can be transformed into each other by a sequence of edge flips.*

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Lawson also proved that any two triangulations of an n-point set can be transformed into each other by a sequence of $O(n^2)$ edge flips, and later, Hurtado et al. proved that the number of edge flips is $\Omega(n^2)$ [5].

Let us consider combinatorial triangulations T of point sets. In this context, the edges of T can be non-crossing curves joining pairs of points in S, which can also be moved around. Wagner [11] proved that any two n-vertex triangulations on the plane can be transformed into each other by a sequence of $O(n^2)$ edge flips, and Komuro improved this result by showing that $O(n)$ edge flips always suffice [6]. This surprising result proves that there is a substantial difference between geometric and combinatorial triangulations, for more details see [4].

A *quadrangulation* of a planar point set S is a geometric plane graph whose vertex set is S, whose outer cycle coincides with $\mathrm{Conv}(S)$, and each of whose finite faces is a quadrilateral. By an easy combinatorial argument, one can prove that if S admits a quadrangulation, then $\mathrm{Conv}(S)$ must have an even number of points of S. It is easy to see that this condition is also sufficient for any planar point set S with $|S| \geq 4$ to have a geometric quadrangulation. (We also know some results on quadrangulations of planar point sets with color constraints and Steiner points [2,3].) However, we know nothing about edge flips in quadrangulations in the geometrical setting. On the other hand, there are many results on edge flips in quadrangulations in the combinatorial settings, see for example [9,10].

In this paper, we deal with a *polygon* $P = v_1 \cdots v_n$ with $n \geq 3$, which is a connected 2-regular geometric plane graph with vertex set $\{v_1, \ldots, v_n\}$. The set $\{v_1, \ldots, v_n\}$ is not necessarily in general position, but we do not allow for interior angles at any vertex of P to be equal to π.

Let $n \geq 4$ be an integer, and let $P = v_1 \cdots v_n$ be an n-sided polygon, which is not necessarily convex. A *geometric quadrangulation* of P is a geometric plane graph obtained from P by adding diagonals to the interior of P so that every finite face is a quadrilateral, and that the boundary of the infinite face coincides with P. For example, in Figure 2(1) we show a geometric quadrangulation of the polygon $P = v_1 \cdots v_{16}$ in which the diagonals we added are represented using dotted line segments. Since every quadrangulation of P is a bipartite graph, we can color the vertices of P black and white alternately, and note that every diagonal must join a black and a white vertex.

A polygon P is *quadrangulatable* if P admits a geometric quadrangulation, and thus it is *even-sided* (i.e., it has an even number of sides). It is easy to see that every n-sided polygon P with $n \geq 3$ admits a geometric *triangulation*, however not every even-sided polygon admits a quadrangulation, a counterex-

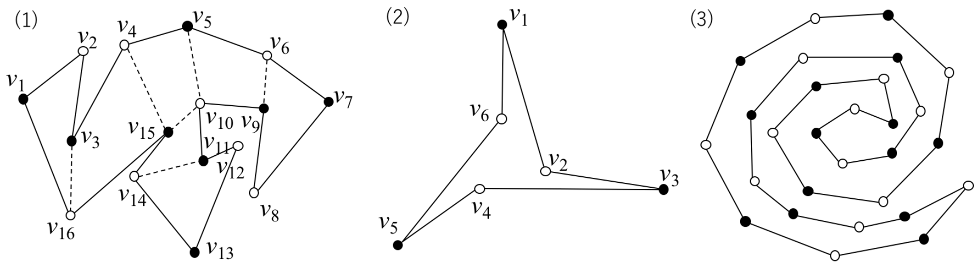

Fig. 2. (1) Geometric quadrangulation of a 16-sided polygon, (2) non-quadrangulatable polygon P_1, (3) polygon of spirality 1

ample P_1 is shown in Figure 2(2); note that the line segments v_1v_4, v_2v_5 nor v_3v_6 are not diagonals of P_1. Our first result is to answer partially the question of which even-sided polygons are quadrangulatable.

For a polygon $P = v_1 \cdots v_n$, we define a new notion, called "the spirality" of P, which measures how close P is to being a convex polygon. We say that v_i is a *concave* (resp., *convex*) vertex of P if the interior angle of v_i in P is greater (resp., less) than π. The interval $[v_p, v_{p+1}, \ldots, v_q]$ is *maximally-concave* if all of $v_p, v_{p+1}, \ldots, v_q$ are concave (possibly $p = q$), but both v_{p-1} and v_{q+1} are convex, where $v_i = v_{n+i}$ for each i. The *spirality* of P, denoted by $\mathrm{sp}(P)$, is the number of distinct maximally-concave intervals in P. By definition, P is convex if and only if $\mathrm{sp}(P) = 0$. (Another notion of how convex a polygon is, called "k-convexity" is defined in [1], but it is different from the spirality of a polygon.)

For example, the two polygons in Figures 2(1) and (2) have spirality 3, since (1) has exactly three maximally-concave intervals $[v_3]$, $[v_9, v_{10}, v_{11}]$ and $[v_{15}]$, and (2) has $[v_2]$, $[v_4]$ and $[v_6]$. Surprisingly, the polygon shown in Figure 2(3) has spirality 1.

The following is our first result.

Theorem 1.3 *Let $n \geq 4$ be an even integer and let P be an n-sided polygon. If $\mathrm{sp}(P) \leq 1$, then P is quadrangulatable. On the other hand, for any large even integer n, there exists an n-sided non-quadrangulatable polygon of spirality* 2.

Figure 3 shows an example of a non-quadrangulatable 6-sided polygon of spirality 2 and a large non-quadrangulatable one of spirality 2 extended from it. We can also construct non-quadrangulatable polygons of large spirality, star-like structures similar to that shown in Figure 2(2).

Let us proceed to our second result. Let P be a polygon with a geometric quadrangulation Q, and let $e = v_av_d$ be a diagonal of Q shared by two quadrilateral faces f_1 and f_2 with $f_1 \cup f_2$ bounded by $v_av_bv_cv_dv_ev_f$. An *edge*

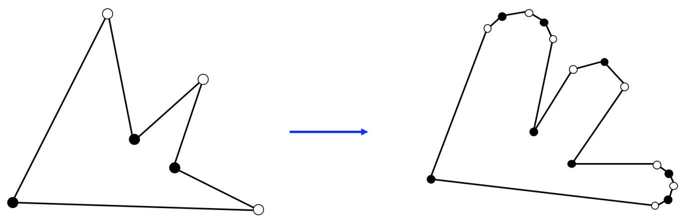

Fig. 3. Extension of a non-quadrangulatable polygon of spirality 2

flip of e in $f_1 \cup f_2$ is to remove e and replace it (if possible) by one of the edges $v_b v_e$, or $v_c v_f$. If e can be flipped in Q, then e is *flippable*. Figure 4 shows two geometric quadrangulations of the same 8-sided polygon, which are related by a single edge flip of the diagonal $v_1 v_6$ to $v_2 v_7$. (In both of them, the diagonal $v_2 v_5$ is not flippable.) Two geometric quadrangulations of P are *flip-equivalent* if they can be transformed into each other by a sequence of edge flips. A geometric quadrangulation Q is *frozen* if every diagonal of Q is not flippable. Note that a frozen geometric quadrangulation of P is not flip-equivalent to any other geometric quadrangulation of P.

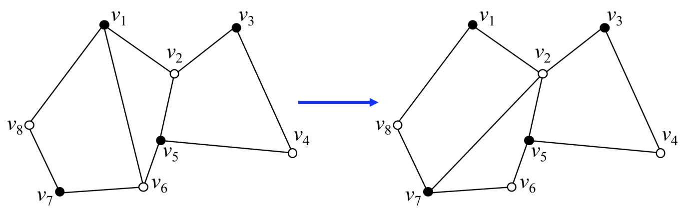

Fig. 4. Edge flip in geometric quadrangulations

The following is our second result on edge flips in geometric quadrangulations on a polygon.

Theorem 1.4 *Let Q_1 and Q_2 be two geometric quadrangulations of the same polygon P. If* $\mathrm{sp}(P) \leq 2$*, then Q_1 and Q_2 are flip-equivalent. On the other hand, there exists a polygon of spirality* 3 *two of whose geometric quadrangulations are not flip-equivalent.*

Figure 5 shows two frozen geometric quadrangulations Q_1 and Q_2 of the same polygon of spirality 3. Since they are not flip-equivalent, the condition "$\mathrm{sp}(P) \leq 2$" in Theorem 1.4 is best possible. It is not so difficult to construct an even-sided polygon of large spirality which admits two distinct frozen geometric quadrangulations.

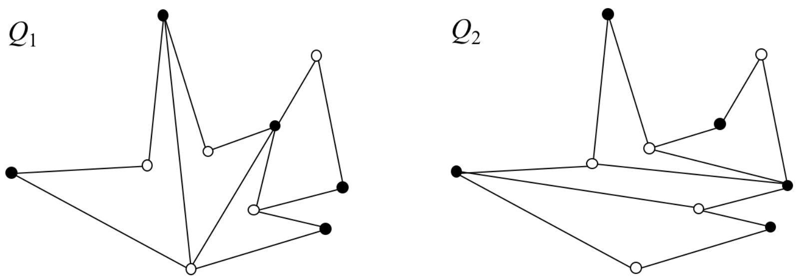

Fig. 5. Non-flip-equivalent geometric quadrangulations of a polygon of spirality 3

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