Sums of finite subsets in $\mathbb{R}^d$

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Abstract

Let $B_1, B_2, \ldots, B_m$ be nonempty finite subsets of $\mathbb{R}^d$ with B_i not contained in an affine hyperplane for each $i \in \{2, 3, \ldots, m\}$. First we get a sharp lower bound on $|B_1 + B_2|$ when $|B_2| = d + 1$. Using this result and other ideas, we find a nontrivial lower bound on $|B_1 + B_2 + \ldots + B_m|$ which generalizes a result of M. Matolcsi and I. Z. Ruzsa [7].

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1 Introduction

For any nonempty subsets A and B of $\mathbb{R}^d$, $n \in \mathbb{N}$ and $\lambda \in \mathbb{R}$, we write

$$\begin{aligned} A+B &:= \{\mathbf{a}+\mathbf{b} :\ \mathbf{a} \in A, \mathbf{b} \in B\} \\ nA &:= \overbrace{A+A+\ldots+A}^{n\text{-times}} \\ -A &:= \{-\mathbf{a} :\ \mathbf{a} \in A\} \\ \lambda \cdot A &:= \{\lambda \mathbf{a} :\ \mathbf{a} \in A\}. \end{aligned}$$

We denote by $\dim A$ the affine dimension of A; thus $\dim A = d$ if and only if A is not contained in an affine hyperplane. The convex hull of A is denoted by $\mathrm{conv} A$.

Let A and B be nonempty finite subsets of $\mathbb{R}^d$. The lower bounds on $|A+B|$ have been widely studied, see [1], [2], [3], [4], [5], [6], [7], [8], [9]. We shall present some new results in this direction.

2 First part

R. J. Gardner and P. Gronchi studied in [4] how to bound $|A+B|$ in terms of $|A|, |B|$ and d; in particular they proved the following statement.

Theorem 2.1 *Let A and B be nonempty finite subsets of $\mathbb{R}^d$. Assume that $|B| \le |A|$ and $\dim B = d$. Then*

$$|A+B| \ge |A| + (d-1)|B| + (|A|-d)^{\frac{d-1}{d}}(|B|-d)^{\frac{1}{d}} - \frac{d(d-1)}{2}.$$

Proof. See [4, Thm. 6.5]. □

For each $m \in \mathbb{N}$, set

$$\sigma(m,1) := \max\left\{ n \in \mathbb{Z} :\ \binom{n+d-1}{d} \le m-1 \right\},$$

and, for all $i \in \{2, 3, \ldots, d\}$,

$$\sigma(m,i) := \max\left\{ n \in \mathbb{Z} :\ \binom{n+d-i}{d-i+1} \le m-1-\sum_{j=1}^{i-1} \binom{\sigma(m,j)+d-j}{d-j+1} \right\};$$

hence the definition of $\sigma(m,d)$ leads to

$$m-1=\sum_{i=1}^{d}\binom{\sigma(m,i)+d-i}{d-i+1}.$$

Theorem 2.2 *Let A and B be nonempty finite subsets of $\mathbb{R}^d$. Assume that $|B|=d+1$ and $\dim B=d$. Then*

$$|A+B|\geq 1+\sum_{i=1}^{d}\binom{\sigma(|A|,i)+d-i+1}{d-i+1}.$$

Theorem 2.2 is optimal in the sense that the equality can be achieved. Moreover, Gardner and Gronchi have shown in [4, p. 4004] that it is impossible to find $c>1$ such that, for all A and B with $\dim A=\dim B=d$, we have the inequality

$$|A+B|\geq c|A|+f_1(|B|)+f_2(d)$$

where f_1 (resp. f_2) depends only on $|B|$ (resp. d). However, it is natural to ask if there is a slow decreasing function $f:\mathbb{R}\to\mathbb{R}$ such that $f(x)>1$ for all $x>0$ and

$$|A+B|\geq f(|A|)|A|+f_1(|B|)+f_2(d).$$

This is the motivation of the first part of our research. The first application of Theorem 2.2 will give an answer to the question asked above when $|B|=d+1$.

Corollary 2.3 *Let A and B be nonempty finite subsets of $\mathbb{R}^d$ and $n\in\mathbb{N}\cup\{0\}$. Assume that $|B|=d+1$ and $\dim B=d$. If $|A|\leq\binom{n+d}{d}$, then*

$$|A+B|\geq\left(1+\frac{d}{n+1}\right)|A|.$$

For any nonempty finite subset A of $\mathbb{R}^d$,

$$\binom{\sigma(|A|+1,1)+d-1}{d}\leq|A|<\binom{\sigma(|A|+1,1)+d}{d}.$$

Now note that for $d>1$,

$$\left(\frac{\sigma(|A|+1,1)+1}{d}\right)^d\leq\binom{\sigma(|A|+1,1)+d-1}{d}\leq|A|;$$

then, for all $d \in \mathbb{N}$,

$$|A| + (|A| - d)^{\frac{d-1}{d}} \leq \left(1 + \frac{d}{\sigma(|A|+1,1)+1}\right)|A|.$$

Therefore Corollary 2.3 provides a better lower bound than the one obtained by Theorem 2.1 when $|B| = d+1$.

3 Second part

Another important result in the area is the following theorem of Matolcsi and Ruzsa which generalizes a famous result of G. A. Freiman [3, Lemma 1.14].

Theorem 3.1 *Let A and B be nonempty finite subsets of $\mathbb{R}^d$. Assume that $\dim B = d$ and $A \subseteq \mathrm{conv} B$. For any $m \in \mathbb{N}$,*

$$|A + (m-1)B| \geq \binom{d+m-1}{m-1}|A| - \binom{d+m-1}{m}(m-1).$$

Proof. See [7, Thm. 1.5]. □

Matolsci and Ruzsa asked in [7, Prob. 1.8] if the lower bound of Theorem 3.1 was still correct for more general subsets of $\mathbb{R}^d$. Using Corollary 2.3 and other ideas, we get the following statement.

Theorem 3.2 *Let $B_1, B_2, \ldots, B_m$ be nonempty finite subsets of $\mathbb{R}^d$ with $\dim B_2 = \dim B_3 = \ldots = \dim B_m = d$. For each $i \in \{1, 2, \ldots, m-1\}$, let $n_i \in \mathbb{N}$ be such that*

$$B_1 + B_2 + \ldots + B_i \subseteq n_i \cdot \mathrm{conv} B_{i+1}.$$

Then

$$|B_1 + B_2 + \ldots + B_m| \geq \left(\prod_{i=1}^{m-1} \frac{n_i + d}{n_i}\right)|B_1| - \sum_{i=1}^{m-1}\left(\frac{\binom{n_i+d}{d-1}}{n_i} \prod_{j=i+1}^{m-1} \frac{n_j+d}{n_j}\right).$$

One application of Theorem 3.2 is the positive answer to the problem proposed by Matolsci and Ruzsa.

Corollary 3.3 *Let $B_1, B_2, \ldots, B_m$ be nonempty finite subsets of $\mathbb{R}^d$ with $\dim B_2 = \dim B_3 = \ldots = \dim B_m = d$. Assume that*

$$B_1 \subseteq \mathrm{conv} B_2 \subseteq \mathrm{conv} B_3 \subseteq \ldots \subseteq \mathrm{conv} B_m.$$

Then

$$|B_1 + B_2 + \ldots + B_m| \geq \binom{d+m-1}{m-1}|B_1| - \binom{d+m-1}{m}(m-1).$$

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