Krein-Milman Spaces

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Abstract

The Krein-Milman theorem characterizes convex subsets in topological vector spaces. Convex geometries were invented as proper combinatorial abstractions of convexity. Further, they turned out to be closure spaces satisfying the Krein-Milman property. Violator spaces were introduced in an attempt to find a general framework for LP-problems. In this work, we investigate interrelations between violator spaces and closure spaces. We prove that a violator space with a unique basis satisfies the Krein-Milman property. Based on subsequent relaxations of the closure operator notion we introduce *convex spaces* as a generalization of violator spaces and extend the Krein-Milman property to uniquely generated convex spaces.

Keywords: Krein-Milman property, convex space, violator space, closure space, convex geometry, antimatroid, extreme point.

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1 Introduction

Intuitively speaking, the Krein-Milman theorem says that every convex set is the convex hull of its extreme points. Originally, this theorem characterized convex subsets in topological vector spaces. In last few decades the theory of convexity was considerably developed and the Krein-Milman theorem was studied for various combinatorial structures: closure spaces, metric spaces, ordered sets and lattices, graphs and so on.

Definition 1.1 Let E be a finite set. $\tau : 2^E \to 2^E$ is a *closure operator on* E if for all subsets $X, Y \subseteq E$ the following properties are satisfied:

C1: $X \subseteq \tau(X)$ (expansivity),
C2: $X \subseteq Y \Rightarrow \tau(X) \subseteq \tau(Y)$ (isotonicity),
C3: $\tau(\tau(X)) = \tau(X)$ (idempotence).

(E, τ) is a *closure space* if τ is a closure operator. A set $A \subseteq E$ is *closed* if $A = \tau(A)$. Clearly, the family of closed sets $K = \{A \in E : A = \tau(A)\}$ is closed under intersection. Conversely, every set system (E, K) closed under intersection is a family of closed sets of the closure operator

$$\tau_K(X) = \cap\{A \in K : X \subseteq A\}.$$

An element x of a subset $X \subseteq E$ is an *extreme point of* X if $x \notin \tau(X - x)$. The set of extreme points of X is denoted by $ex(X)$.

A closure space (E, τ) satisfies the *Krein-Milman property* if for every closed set

$$A \subseteq E : A = \tau(ex(A)).$$

Such closure spaces are known as *convex geometries* [5]. They were originally invented by Edelman and Jamison in 1985 as proper combinatorial abstractions of convexity [2]. The convex hull operator on the Euclidean space E^n is a canonical example of a closure operator defining a convex geometry.

Definition 1.2 [3] A *violator space* is a pair (E, ν), where E is a finite set and ν is a mapping $2^E \to 2^E$ such that for all subsets $X, Y \subseteq E$ the following properties are satisfied:

V1: $X \cap \nu(X) = \emptyset$ (consistency),
V2: $(X \subseteq Y$ and $Y \cap \nu(X) = \emptyset) \Rightarrow \nu(X) = \nu(Y)$ (locality).

Violator spaces have been introduced and analyzed as a combinatorial framework that encompasses Linear Programming (LP) and other geometric optimization problems [3,7]. Originally, violator spaces were defined for set of constraints E, where each subset of constraints $X \subseteq E$ is associated with

$\nu(X)$ - the set of all constraints violating X. For instance, the problem of computing the smallest enclosing ball of a finite set of points in R^d is an LP-type problem. Here, the set E is a set of points in R^d, and the violated constraints of some subset of the points X are exactly the points lying outside the smallest enclosing ball of X.

Let $\alpha, \beta : 2^E \to 2^E$ be two operators satisfying $X \subseteq \alpha(X)$ and $\beta(X) \subseteq X$. We define (E, α, β) as a *Krein-Milman space* if for every set

$$X \subseteq E : \alpha(X) = \alpha(\beta(X)).$$

A well-known example of a Krein-Milman space is a convex geometry ($\alpha = \tau$, $\beta = ex$). One of our main findings is the characterization of Krein-Milman violator spaces.

2 Violator mappings and closure operators

Proposition 2.1 *[4] Let (E, τ) be a closure space. Define $\nu(X) = E - \tau(X)$. Then (E, ν) is a violator space.*

Proposition 2.2 *[4] Let (E, ν) be a violator space. Define $\varphi(X) = E - \nu(X)$. Then the operator φ satisfies expansivity and idempotence.*

In what follows, if (E, ν) is a violator space and $\varphi(X) = E - \nu(X)$, then (E, φ) will be called a violator space as well.

Every violator space (E, φ) satisfies the following property [4]

(1) $$(X \subseteq Y \subseteq Z) \wedge (\varphi(X) = \varphi(Z)) \Rightarrow \varphi(X) = \varphi(Y) = \varphi(Z).$$

Since the property deals with all sets lying between two given sets, following [6] we say that an operator is *convex* if it satisfies (1). Notice that locality is equivalent to the following property that we call *self-convexity* by analogy with convexity

C22 : $(X \subseteq Y \subseteq \varphi(X)) \Rightarrow \varphi(X) = \varphi(Y)$.

One can see that while closure spaces satisfy expansivity, isotonicity, and idempotence, violator spaces satisfy expansivity, self-convexity, and idempotence, and so may be considered as *weak closure spaces*.

Moreover, there are violator spaces, where the operator $\varphi(X)$ does not satisfy isotonicity [3]. Hence, there exists a violator space which is not a closure space.

It is easy to check, that expansivity and self-convexity imply convexity, while convexity and idempotence imply self-convexity. The following example shows that convexity accompanied with expansivity does not obligate a space to be a violator space. Let $E = \{1, 2, 3\}$. Define $\delta(X) = X$ for each $X \subseteq E$

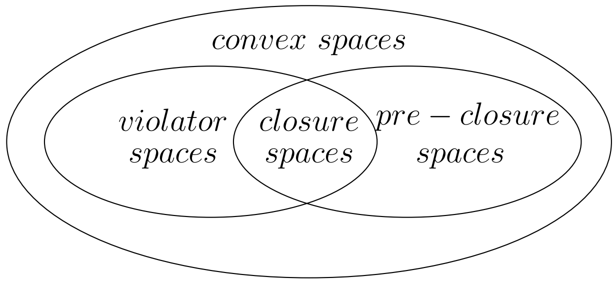
except $\delta(\{1\}) = \{1,2\}$, and $\delta(\{1,2\}) = \{1,2,3\}$. It is easy to see that the mapping δ satisfies expansivity and convexity. At the same time, it does not satisfy neither self-convexity, nor idempotence.

$$\{1\} \subseteq \{1,2\} \subseteq \delta(\{1\}), \delta(\{1,2\}) = \{1,2,3\} \neq \delta(\{1\}),$$

and $\delta(\delta(\{1\})) \neq \delta(\{1\})$.

Definition 2.3 A space (E, δ) is *convex* if $\delta : 2^E \to 2^E$ satisfies expansivity and convexity.

The last example shows that there exists a convex space, which is not a violator space. Another example of the operator that does not satisfy idempotence is a *pre-closure operator* - the operator that satisfies only expansivity and isotonicity. It is easy to check that pre-closure operators satisfy the convexity, and so every pre-closure space is convex.



3 Uniquely generated spaces and the Krein-Milman property

Let we have an arbitrary space (E, α) with the mapping $\alpha : 2^E \to 2^E$. We say that $B \subseteq E$ is a *generator* of $X \subseteq E$ if $\alpha(B) = \alpha(X)$. For $X \subseteq E$, a *basis (minimal generator) of* X is a inclusion-minimal set $B \subseteq E$ (not necessarily included in X) with $\alpha(B) = \alpha(X)$. A space (E, α) is *uniquely generated* if every set $X \subseteq E$ has a unique basis. A well-known characterization of closure operators states equivalence between uniqueness of the basis and the Krein-Milman property [5]. We extend this property to some more spaces.

Proposition 3.1 *The following assertions are equivalent:*

(i) The space (E, α) *is uniquely generated;*

(ii) For every set $X \subseteq E$ *its bases are included in the intersection of all the generators of* X*:* $B \subseteq \bigcap\{Y : \alpha(Y) = \alpha(X)\}$*;*

(iii) For every set $X \subseteq E$ *its bases are subsets of* X*.*

A convex space (E, α) is uniquely generated if and only if for every

$$X, Y \subseteq E : \alpha(X) = \alpha(Y) \Rightarrow \alpha(X \cap Y) = \alpha(X) = \alpha(Y).$$

We can rewrite this property as follows. For every set $X \subseteq E$ of a uniquely generated convex space (E, α), the basis B of X is an intersection of all generators of X:

$$B = \bigcap \{Y \subseteq E : \alpha(Y) = \alpha(X)\}. \tag{2}$$

Following the definition of extreme points in closure spaces we define

$$ex(X) = \{x \in X : x \notin \alpha(X - x)\}.$$

For violator spaces, it gives us:

$$x \notin \alpha(X - x) \Leftrightarrow \alpha(X) \neq \alpha(X - x),$$

and consequently:

$$x \in ex(X) \Leftrightarrow \alpha(X) \neq \alpha(X - x).$$

For convex spaces, the implication is only one-sided:

$$x \in ex(X) \Rightarrow \alpha(X) \neq \alpha(X - x).$$

Another approach to extreme points reads as follows:

$$\bar{ex}(X) = \{x \in X : \alpha(X) \neq \alpha(X - x)\}.$$

It is worth notice that $ex(X) = \bar{ex}(X)$ for violator spaces, while for convex spaces only the inclusion $ex(X) \subseteq \bar{ex}(X)$ may be claimed.

Proposition 3.2 *Let (E, α) be a convex space. Then $x \in \bar{ex}(X)$ if and only if x belongs to every generator (and so, to every basis) of X contained in X.*

Corollary 3.3 (i) *If (E, α) is a violator space, then*

$$ex(X) = \bigcap \{B \subseteq X : \alpha(B) = \alpha(X)\}. \tag{3}$$

(ii) *If (E, α) is a convex space, then*

$$\bar{ex}(X) = \bigcap \{B \subseteq X : \alpha(B) = \alpha(X)\}. \tag{4}$$

If α satisfies idempotence, then each generator of X is a generator of $\alpha(X)$ as well. Hence, for violator spaces, we have

$$ex(\alpha(X)) = \bigcap \{B \subseteq \alpha(X) : \alpha(B) = \alpha(X)\} =$$

$$ex(X) \cap \bigcap \{B \subseteq \alpha(X) \wedge B \nsubseteq X : \alpha(B) = \alpha(X)\} \subseteq ex(X). \tag{5}$$

In particular, $ex(\alpha(X)) \subseteq X$, which may be considered as a combinatorial interpretation of Milman's theorem [1].

Theorem 3.4 *Let (E, α) be a violator space. Then (E, α) is uniquely generated if and only if for every set $X \subseteq E$, $\alpha(X) = \alpha(ex(X))$.*

For a uniquely generated violator space each basis of a set is contained in this set (Proposition 3.1). Therefore, the inclusion $ex(\alpha(X)) \subseteq ex(X)$ (5) turns out to be the equality $ex(\alpha(X)) = ex(X)$. Thus we conclude with the following.

Corollary 3.5 *Let (E, α) be a uniquely generated violator space. Then $\alpha(X) = \alpha(ex(X))$ and $ex(X) = ex(\alpha(X))$.*

Proposition 3.6 *Let (E, α) be a uniquely generated convex space. Then for every set $X \subseteq E$, $\alpha(X) = \alpha(\bar{ex}(X))$.*

In summary, the proper choice of the operator β allows to interpret uniquely generated convex spaces and uniquely generated violator spaces as Krein-Milman spaces.

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