An improvement of the lower bound on the maximum number of halving lines in planar sets with 32 points

Javier Rodrigo [1]

Departamento de Matemática Aplicada. E.T.S. de Ingeniería.
Universidad Pontificia Comillas de Madrid
Madrid, Spain

Mª Dolores López [2]

Departamento de Matemática e Informática Aplicadas a las Ingenierías Civil y Naval de la E.T.S.I. Caminos, Canales y Puertos
Universidad Politécnica de Madrid
Madrid, Spain

Abstract

In this paper we give a recursive lower bound on the maximum number of halving lines for sets in the plane and as a consequence we improve the current best lower bound on the maximum number of halving lines for sets in the plane with 32 points.

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1 Introduction

The search for upper and lower bounds for the maximum number of halving lines in sets of n points of the plane, is a challenging task in the existing Combinatorial Geometry literature due to its relation with the rectilinear crossing number problem[1], [4].

[1] Email: jrodrigo@comillas.edu
[2] Email: marilo.lopez@upm.es

An example of this relation is a conjecture by Aichholzer et al. [3] that says that every set attaining the rectilinear crossing number maximizes the number of halving lines.

Roughly speaking, a halving line of a finite planar set P is a line passing by two points of P that splits P in two equally sized subsets (see definitions 1.1, 1.2 below for a more formal definition). With respect to the lower bounds for the maximum number of these halving lines in sets of n points for even n (h_n), we have that $h_n \geq \frac{n}{4}\log_2\left(\frac{n}{3}\right)$, given by Erdõs, Lovász, Simmons and Strauss in 1973 [6]. Later, this bound was improved to $h_n \geq n\log_4\left(\frac{2\,n}{3}\right)$ by Eppstein [5]. More recently, Tóth[8] found a better asymptotic lower bound : $h_n \geq \frac{n}{2}e^{0.744\sqrt{\log\left(\frac{n}{2}\right)}-2.7}$. Nivasch[7] improved the constant in the exponent and found the up-to-date best asymptotic lower bound : $h_n \geq Cn\frac{e^{0.744\sqrt{\log(4)\log(n)}}}{\sqrt{\log(n)}}$ for some fixed constant C.

For small n, the exact value of h_n is known for $n \leq 27$. For $28 \leq n \leq 32$, there are small gaps between the best lower bound and the best upper bound of h_n.

In this paper, we improve by one unit the current best lower bound of h_n for $n = 32$ reported in [2],[1]. Concretely, we prove that $h_{32} \geq 74$. This reduces the gap with the up-to-date best upper bound of h_{32}, namely $h_{32} \leq 79$ [1]. This also yields a reduction by one unit of the current best upper bound of the rectilinear crossing number for sets of 32 points ($cr(32)$, see definition 1.3 below), assuming that the aforementioned conjecture is true.

The formal definition of a halving line is as follows:

Definition 1.1 Given a set of points in the plane $A = \{p_1, \ldots, p_n\}$, a k-edge of A is a line R that joins two points of A and leaves k points of A in one of the open half planes (that is to say, in one connected component of $\Re^2 - R$).

Definition 1.2 Given a set of points in the plane $A = \{p_1, \ldots, p_n\}$, a halving line of A is an $\left[\dfrac{n-2}{2}\right]$-edge of A.
Remark: For even n, a halving line of A leaves the same number of points of A in each half plane: $\dfrac{n-2}{2}$.

We also give the following definitions:

Definition 1.3 Given a finite set of points in the plane P, assume that we join each pair of points of P with a straight line segment. The rectilinear crossing number of P ($cr(P)$) is the number of segment-crossings. The rectilinear crossing number of n ($cr(n)$) is the minimum of $cr(P)$ over all the sets P with n points.

Definition 1.4 Given a finite set of points in the plane $P = \{p_1, \ldots, p_n\}$, the subjacent graph of P is the graph $G = (V, E)$ such that $V = P$ and $\{p_i, p_j\} \in E$ if the line joining p_i, p_j is a halving line of P (sometimes we will identify the edge $\{p_i, p_j\}$ with the halving line of P containing p_i, p_j. We will call this line $p_i - p_j$). The valence of a vertex is the number of edges incident with the vertex in the subjacent graph.

The outline of the rest of the paper is as follows: In Section 2 we give a technical lemma, in Section 3 we apply said lemma to state the main result, the aforementioned improvement of the lower bound and in Section 4 we give some concluding remarks.

2 A technical Lemma

For the following lemma we need two previous definitions:

Definition 2.1 Let A be a finite set with $|A| = n$, n an even number, $Hal = \{\text{halving lines of A}\} = \{R_1, \ldots, R_m\}$ and let R be a line such that $R_i \cap R \neq \emptyset$ and $R_i \cap R \neq R_j \cap R$ for $i \neq j$, $i, j \in \{1, \ldots, m\}$.

The arrangement of the elements of Hal with respect to R is the order according to the y coordinate of the intersection points of R and R_i, $i = 1, \ldots, m$. That is to say, $R_i \leq R_j$ if $y_i \leq y_j$, being $R_i \cap R = \{(x_i, y_i)\}$, $R_j \cap R = \{(x_j, y_j)\}$.

Definition 2.2 Let R be a line and let p_i, p_j be two points that do not belong to R, we say that R separates p_i, p_j if it does not leave p_i, p_j in the same open half plane.

Lemma 2.3 *Consider a set $Q = \{p_1, \ldots, p_n\}$ where n is an even number, $n > 2$, let R' be an $\frac{n-4}{2}$-edge of Q and let R be a line satisfying the following conditions: (a) R meets all the halving lines of Q at different points; (b) R separates the points of Q in the first and last halving lines that it crosses with the order of Definition 2.1; (c) R is parallel to R' and it is contained in the half plane generated by R' with $\frac{n-4}{2}$ points of Q.*

Then we can locate two points p_{n+1}, p_{n+2} in such a way that $P := Q \cup \{p_{n+1}, p_{n+2}\}$ satisfies $h(P) \geq h(Q) + 5$.

Proof. Let $R_1 = p_i - p_j$, $R_2 = p_k - p_l$ respectively be the maximum and minimum of $Hal = \{\text{halving lines of A}\} = \{R_1, \ldots, R_m\}$ with the order of

Definition 2.1. Then R separates p_i, p_j and also separates p_k, p_l as per condition (b) above.

If we consider two points p_{n+1}, p_{n+2} in R satisfying

(i) The y coordinate of p_{n+1} is lower than the one of $R_1 \cap R$ and higher than the one of $R_i \cap R$, $\forall i \neq 1$.

(ii) The y coordinate of p_{n+2} is higher than the one of $R_2 \cap R$ and lower than the one of $R_i \cap R$, $\forall i \neq 2$.

Then the number of halving lines of the set $P = Q \cup \{p_{n+1}, p_{n+2}\}$ is greater than or equal to the number of halving lines of Q separating p_{n+1}, p_{n+2} plus the number of halving lines of Q containing to p_{n+1} or p_{n+2} plus one because R' is a halving line of P as per condition (c) above.

We have that every halving line of Q except for R_1, R_2 separates p_{n+1}, p_{n+2} , so the number of halving lines of Q separating p_{n+1}, p_{n+2} is $h(Q) - 2$.

Now, if we have located p_{n+1}, p_{n+2} close enough to R_1, R_2, then we can get that $p_i - p_{n+1}$, $p_j - p_{n+1}$, $p_k - p_{n+2}$, $p_l - p_{n+2}$ are halving lines of P.

Therefore, since $p_{n+1} - p_{n+2}$ is not a halving line of P because it is parallel to R' by hypothesis, we have two more halving lines of P including p_{n+1} or p_{n+2} to ensure that the degrees of p_{n+1}, p_{n+2} in the underlying graph are odd numbers, and then $h(P) \geq (h(Q) - 2) + 6 + 1 = h(Q) + 5$ as desired. □

3 The improvement of the lower bound

In this section we show examples of sets that shift the lower bound of h_{32}. They are constructed by adding two points to a set Q that gives the current best lower bound of h_{30}, in such a way that every halving line but two of Q are preserved and seven new halving lines are created by following the procedure of the proof of Lemma 2.3. The proof of the following proposition gives a first example:

Proposition 3.1 *It is satisfied that $h_{32} \geq 74$.*

Proof. We consider as basis set, the set $Q = \{p_1, \ldots, p_{30}\}$ with 30 points included in [2], and we find a line R satisfying the conditions of Lemma 2.3 (for instance, a parallel line to $R' = p_2 - p_{18}$). Then we add two points p_{31}, p_{32} according to the proof of Lemma 2.3 to get a set $P = \{p_1, \ldots, p_{32}\}$ such that $h(P) \geq h(Q) + 5 = 74$. The coordinates of the points of P are:

$$P = \left\{ \begin{array}{l} v_1 = (9259, 16958)\,,\, v_2 = (9763, 16199)\,,\, v_3 = (9977, 16397)\,, \\ v_4 = (10248, 16225)\,,\, v_5 = (10666, 16385)\,,\, v_6 = (12849, 16335)\,, \\ v_7 = (18577, 16451)\,,\, v_8 = (10391, 16281)\,,\, v_9 = (28477, 16613)\,, \\ v_{10} = (15909, 16415)\,,\, v_{11} = (9446, 15905)\,,\, v_{12} = (9540, 16541)\,, \\ v_{13} = (9262, 16627)\,,\, v_{14} = (9282, 16947)\,,\, v_{15} = (8912, 17261)\,, \\ v_{16} = (7842, 19232)\,,\, v_{17} = (5141, 23755)\,,\, v_{18} = (9154, 17055)\,, \\ v_{19} = (0, 32394)\,,\, v_{20} = (6820, 20291)\,,\, v_{21} = (9949, 16415)\,, \\ v_{22} = (9355, 16177)\,,\, v_{23} = (9419, 15893)\,,\, v_{24} = (9146, 15771)\,, \\ v_{25} = (9075, 15320)\,,\, v_{26} = (7921, 13407)\,,\, v_{27} = (5206, 8451)\,, \\ v_{28} = (9121, 15603)\,,\, v_{29} = (480, 0)\,,\, v_{30} = (6432, 10637)\,, \\ v_{31} = (8525, 17938.4)\,,\, v_{32} = (9855, 16069) \end{array} \right\}$$

This implies that $h_{32} \geq 74$ as desired. □

Remark: Since the set included in [2] that gives $cr(32) \leq 12836$ has 73 halving lines, we have that if the conjecture of [3] was true, then Proposition 3.1 would imply that $cr(32) \leq 12835$.

4 Conclusions

We have considered the task of sharpening the best lower bound for the maximum number of halving lines in planar sets of 32 points. We have achieved it by means of an example that improve by one unit the current best lower bound for these sets. This carries with it either the improvement of the upper bound of the rectilinear crossing number for planar sets of 32 points or the refutation of a conjecture included in [3].

A future line of work could be the search of similar examples for $n = 28, 30$ by means of an improvement of Lemma 2.3.

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