Graphs preserving total distance upon vertex removal

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Abstract

The total distance or Wiener index $W(G)$ of a connected graph G is defined as the sum of distances between all pairs of vertices in G. In 1991, Šoltés posed the problem of finding all graphs G such that the equality $W(G) = W(G - v)$ holds for all their vertices v. Up to now, the only known graph with this property is the cycle C_{11}. Our main object of study is a relaxed version of this problem: Find graphs for which total distance does not change when a particular vertex is removed. We show that there are infinitely many graphs that satisfy this property. This gives hope that Šoltes's problem may have also some solutions distinct from C_{11}.

Keywords: total distance, transmissiom, unicyclic graph, pendant vertex, induced subgraph

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1 Introduction

Average distance is one of the three most robust measures of network topology, along with its clustering coefficient and its degree distribution. Nowadays it has been frequently used in sociometry and the theory of social networks [3]. Wiener index, defined as the sum of distances between all (unordered) pairs of vertices in a graph, besides its crucial role in the calculation of average distance, is the most famous topological index in mathematical chemistry. It is named after Wiener [9], who introduced it in 1947 for the purpose of determining boiling points of alkanes. Since then Wiener index has become one of the most frequently used topological indices in chemistry, since molecules are usually modelled by undirected graphs. Other applications of this graph invariant can be found in crystallography, communication theory and facility location. Wiener index has also been studied in pure mathematics under various names: the gross status, the distance of a graph, the transmission of a graph etc. It seems that the first mathematical paper on Wiener index was published in 1976 [2]. Since then, a lot of mathematicians have studied this quantity very extensively. A great deal of knowledge on Wiener index is accumulated in survey papers [1,6,10]. Nowadays it has been frequently used in sociometry and the theory of social networks [3].

Throughout this paper all graphs will be finite, simple and undirected.

The *total distance* or *Wiener index* $W(G)$ of a connected graph G is defined as the sum of distances between all (unordered) pairs of vertices in G:

$$W(G) = \sum_{\{u,v\} \subseteq V(G)} d_G(u,v) = \frac{1}{2} \sum_{v \in V(G)} t_G(v), \tag{1}$$

where the *distance* $d_G(u,v)$ between vertices u and v is defined as the number of edges on a shortest path connecting these vertices in G, and the distance, or transmission, $t_G(v)$ of a vertex $v \in V(G)$ is the sum of distances between v and all other vertices of G.

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In 1991, Šoltés [8] posed the following problem:

Problem 1.1 *Find all such graphs G that the equality $W(G) = W(G - v)$ holds for all their vertices v.*

Till now, only one such graph is known: it is a cycle with 11 vertices.

Motivated by Šoltés's problem we construct an infinite family of unicyclic graphs which preserve total distance after removal of a particular vertex. In fact, we prove that there are infinitely many unicyclic graphs with this property even when we fix the length of the cycle. Then we show that for every graph G there are infinitely many graphs H such that G is an induced subgraph of H and $W(H) = W(H - v)$ for some vertex $v \in V(H) \setminus V(G)$. Our research is further extended to graphs in which vertex v is of arbitrary degree. For $k \geq 3$ we show that there are infinitely many graphs G with a vertex v of degree k for which $W(G) = W(G - v)$. Moreover, we prove the existence of such graphs when the degree is $n - 1$ or $n - 2$. Finally, we show that dense graphs cannot be a solution of Problem 1.1.

Our contribution shows that the class of graphs, which total distance does not change when a particular vertex is removed, is rich. This gives hope that Šoltes's problem may have another solution beside C_{11}.

2 Results for unicyclic graphs

Theorem 2.1 *Let $c \geq 5$. There exists infinitely many unicyclic graphs G with a cycle of length c for which equality $W(G) = W(G - v)$ holds for some $v \in V(G)$.*

Proof. Our construction of unicyclic graphs G for which $W(G) = W(G - v)$ goes in the following way. Let C_c be a cycle of length c. We denote its vertices consecutively by $v_0, v_1, \ldots, v_{c-1}$. We add to C_c a pendant vertex, to obtain a new graph, then we add another pendant vertex (which may be connected to previously added vertex) and so on, until we get a unicyclic graph G with $W(G) = W(G - v_0)$. Then we continue with adding pendant vertices to create infinitely many graphs G with the property $W(G) = W(G - v_0)$. Of course, since $G - v_0$ has to be connected, we cannot add pendant vertices to v_0. In fact, most of our graphs are obtained from C_c by adding a path to v_{c-1} and a tree to v_1, that is, usually the vertices $v_2, v_3, \ldots, v_{c-2}$ will all have degree 2 in G. $\square$

By studying the case when $c \in \{3, 4\}$, we conclude that there is no unicyclic graph G with a cycle of length c satisfying $W(G) = W(G - v)$ for some

$v \in V(G)$. Moreover, we show that a unicyclic graph G on n vertices for which $W(G) = W(G - v)$ exists if and only if $n \geq 9$.

3 Induced subgraphs

For arbitrary graph G, we prove the existence of infinitely many connected graphs H, containing G as an induced subgraph, and such that $W(H) = W(H - v_0)$ for some vertex $v_0 \in V(H) - V(G)$. The main tool is the following theorem.

Theorem 3.1 [7] *Let G_u and G_v be two graphs with n_u and n_v vertices, respectively, and let $u \in V(G_u)$, $v \in V(G_v)$. If G arises from G_u and G_v by identifying u and v, then*

$$W(G) = W(G_u) + W(G_v) + (n_u - 1)t_{G_v}(v) + (n_v - 1)t_{G_u}(u). \tag{2}$$

We use this result so that one graph contains G as an induced subgraph and the other graph is a cycle.

4 Vertex of a fixed degree

Our first observation is that if a vertex v has degree 1 in G, then $W(G) > W(G-v)$. Since case $d_G(v) = 2$ was already studied through unicyclic graphs, see [4], we focus on v such that $d_G(v) \geq 3$. Our main result is the following theorem.

Theorem 4.1 *For every $k \geq 3$ there exist infinitely many graphs G with vertex v such that $d_G(v) = k$ and $W(G) = W(G - v)$.*

Proof. In each case we show the existence of a graph G_1 with a vertex v such that $d_{G_1}(v) = k$ and $W(G_1) = W(G_1 - v)$. Then we construct an infinite class of graphs by attaching to G_1 a new graph G_2 according to Theorem 3.1, and by taking into a consideration necessary and sufficient condition under which the resulting graph H satisfies $W(H) = W(H - v)$. □

Let $n \geq 7$. When v is of degree close to the order n, $d(v) = n-1$ or $d(v) = n - 2$, we can construct at least one graph G such that $W(G) = W(G - v)$.

4.1 Graphs with large minimum degree

At last, we prove that dense graphs cannot be particular solutions of Problem 1.1. By dense graphs we mean those n-vertex graphs in which the minimum degree $\delta(G)$ is at least $n/2$. Our result relies on the observation that for $n \geq 3$ and $\delta(G) \geq n/2$, we have $\mathrm{diam}(G) \leq 2$.

5 Concluding remarks

It seems rather difficult to find graphs other than C_{11} which are solutions of Šoltes's problem. However, by studying a relaxed version of the problem or by focusing on some particular classes of graph, one could get a better insight into the original problem and find one more soluton of it, or show that such graphs do not exist.

One can consider regular graphs. Note that asking for a graph to be vertex-transitive will be a as well a solution of the Šoltés's problem.

We can pose the following problems.

Problem 5.1 *Are there k-regular connected graphs G other than C_{11} for which the equality $W(G) = W(G - v)$ holds for at least one vertex $v \in V(G)$?*

We know that there are no such graphs for $k \geq n/2$.

One can go further and study graphs G for which equation $W(G) = W(G - S)$ holds for a subset S of the vertex set $V(G)$ consisting of at least 2 vertices.

Problem 5.2 *Find connected graphs G for which*

$$W(G) = W(G - S)$$

for any $S \subset V(G)$, with $|S| \geq 2$.

Our results show the existence of an infinite class of graphs G for which $W(G) = W(G - v)$ for a particular vertex v. It is natural to formulate the following conjecture.

Problem 5.3 *For a given r, find (infinitely many) graphs G for which*

$$W(G) = W(G - v_1) = W(G - v_2) = \cdots = W(G - v_r)$$

for any distinct vertices $v_1, \ldots, v_r \in V(G)$.

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