On characterizing the extreme points of the generalized transitive tournament polytope

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Abstract

A non-negative $n \times n$ matrix $[x_{ij}]$ is called generalized tournament, denoted GTT(n), if: $x_{ii} = 0$ (for all i), $x_{ij} + x_{ji} = 1$ (for all(i,j) with $i \neq j$) and $1 \leq x_{ij} + x_{jk} + x_{ki} \leq 2$ (for all (i,j,k) with i, j, k pairwise distinct). In [9], using hypergraphs associated with GTT matrices, it has been shown that for $n \leq 6$ all the vertices of the GTT(n) polytope are half-integral. In this work, we show that these matrices belong to the class of 2-regular matrices and highlight the related optimization implications. Finally, based on our approach and known partial results, conjectures on characterizing the extreme points of the GTT(n) polytope for $n \geq 7$ are provided.

Keywords: generalized transitive tournaments, k-regular matrices, integral polytopes.

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1 Introduction

A non-negative matrix $X = [x_{ij}]$ of order n satisfying the following two conditions, for all distinct $i, j, k = 1, \ldots, n$:

$$x_{ii} = 0 \tag{1}$$

$$x_{ij} + x_{ji} = 1 \tag{2}$$

is called a *generalized tournament matrix*.

In case X satisfies also that (for all distinct i, j, k):

$$1 \leq x_{ij} + x_{jk} + x_{ki} \leq 2 \tag{3}$$

then it is called *generalized transitive tournament matrix*, abbreviated GTT(n) matrix. The convex polytope defined by (1) – (3) will be denoted by $\mathcal{T}_n$ and a GTT(n) matrix will be said to be an *extreme GTT(n)* matrix provided its entries correspond to an extreme point of $\mathcal{T}_n$.

The problem of characterizing the extreme points of the generalized transitive tournament problem has only been resolved for $n \leq 6$. In particular, it is known that all extreme GTT(n) matrices for $n \leq 5$ are $\{0, 1\}$-matrices, i.e. each element is either 0 or 1, (see [4,5]). Recently, in [9], it was shown that the all the vertices of $\mathcal{T}_6$ are half-integral, i.e. the extreme GTT(6) matrices have elements in $\{0, 1, \frac{1}{2}\}$. In order to prove this result, the authors employed theory from the hypergraphs area and associated them with GTT(n) matrices. We employ a totally different approach in our work showing that the set of extreme GTT(n) matrices for $n \leq 6$ are the vertices of polytopes of the form $\{x : Ax \leq b, x \geq 0\}$ where b is an integral vector and A is a 2-regular matrix. Finally, we conjecture that the extreme GTT(7) matrices are also the vertices of polytopes defined by 2-regular matrices while, for $n > 7$, based on partial results of [3,8] we conjecture that the corresponding extreme GTT(n) matrices are also the extreme points of a polytope of the form $\{x : Ax \leq b, x \geq 0\}$ where A is k-regular with $k > 2$.

In the next section, we provide some definitions and known results which would be needed in Section 3 presenting the main results of this work. In the final section of this work (Section 4), conclusions and open problems are provided.

2 Preliminaries

In matrix form, the equations defined by (1) will be written as $A_1^{(n)} x^{(n)} = b_1^{(n)}$ and, accordingly, we shall write $A_2^{(n)} x^{(n)} = b_2^{(n)}$ in order to represent in matrix form the equalities of (2). Moreover, in order to describe in matrix form the

"$\leq$" and "$\geq$" inequalities of (3) we shall use the notation $b_3^{(n)} \leq A_3^{(n)} x^{(n)}$ and $A_3^{(n)} x^{(n)} \geq b_4^{(n)}$, respectively. Let also :

$$\bar{A}^{(n)} = \begin{bmatrix} A_1^{(n)} \\ A_2^{(n)} \\ A_3^{(n)} \\ A_4^{(n)} \end{bmatrix}, A^{(n)} = \begin{bmatrix} A_2^{(n)} \\ A_4^{(n)} \end{bmatrix} \text{ and } b^{(n)} = \begin{bmatrix} b_2^{(n)} \\ b_4^{(n)} \end{bmatrix}.$$

It should be noted that in what follows, in order to simplify the notation, the (n) will be omitted whenever its value is clear by the context.

There are matrices of special structure which guarantee integral polyhedra. To be more specific, we focus on totally unimodular (TU) matrices for which we have the following well-known result (see e.g. Proposition 2.3 in [7]):

Theorem 2.1 *If A is TU, if b, b', d and d' are integral and if $P(b, b, d, d') = \{x \in \mathbb{R}^n : b' \leq Ax \leq b, d' \leq x \leq d\}$ is not empty, then $P(b, b', d, d')$ is an integral polyhedron.*

k-regular matrices furnish a natural generalization of TU matrices. One important polyhedral results is provided in the following theorem of [1] and its optimization consequences may be found in various fields (see e.g.[2,6]).

Theorem 2.2 *Let A be an integral matrix of size $m \times n$. Then the polyhedron $\{x : c \leq x \leq d, a \leq Ax \leq b\}$ is integral for all vectors $a, b \in k\mathbb{Z}^m$ and $c, d \in k\mathbb{Z}^m$, if and only if A is k-regular.*

3 2-Regularity and the extreme GTT(n) matrices ($n \leq 6$)

3.1 The GTT(4) and GTT(5) cases

Lemma 3.1 *Every extreme $GTT(n)$ $(n = 4, 5)$ matrix is integral.*

Sketch of Proof: For $n = 4$, the equalities $A_2 x = b_2$ and the inequalities $A_4 x = b_4$ are given below in the leftmost system. By subtracting equalities of $A_2 x = b_2$ from inequalities in the $A_4 x = b_4$, the rightmost system of equalities and inequalities is obtained. Clearly, the polyhedron $P = \{x | A_2 x = b_2, A_4 x = b_4, x \geq 0\} = \{x | A_2 x = b_2, A_4' x = b_4, x \geq 0\}$ [2]

[2] Row equivalence of $[A|b]$ and $[A'|b']$ can be easily verified via various free or commercial software packages.

$$
\begin{array}{c}
A_2 \\
\hline
A_4
\end{array}
\underbrace{\left[\begin{array}{cccccccccccc}
1&0&0&1&0&0&0&0&0&0&0&0\\
0&1&0&0&0&0&1&0&0&0&0&0\\
0&0&1&0&0&0&0&0&0&1&0&0\\
0&0&0&0&1&0&0&1&0&0&0&0\\
0&0&0&0&0&1&0&0&0&0&1&0\\
0&0&0&0&0&0&0&0&1&0&0&1\\
\hline
1&0&0&0&1&0&1&0&0&0&0&0\\
0&1&0&1&0&0&0&1&0&0&0&0\\
0&1&0&0&0&0&0&0&1&1&0&0\\
0&0&1&0&0&0&1&0&0&0&0&1\\
1&0&0&0&0&1&0&0&0&1&0&0\\
0&0&1&1&0&0&0&0&0&0&1&0\\
0&0&0&0&1&0&0&0&1&0&1&0\\
0&0&0&0&0&1&0&1&0&0&0&1
\end{array}\right]}_{A}
\underbrace{\left[\begin{array}{c}
x_{12}\\x_{13}\\x_{14}\\x_{21}\\x_{23}\\x_{24}\\x_{31}\\x_{32}\\x_{34}\\x_{41}\\x_{42}\\x_{43}
\end{array}\right]}_{x}
\begin{array}{c} = \\ \hline \leq \end{array}
\underbrace{\left[\begin{array}{c}
1\\1\\1\\1\\1\\1\\ \hline 2\\2\\2\\2\\2\\2\\2\\2
\end{array}\right]}_{b}
\begin{array}{c} b_2 \\ \\ b_4 \end{array}
\sim
\begin{array}{c}
A_2 \\
\hline
A'_4
\end{array}
\underbrace{\left[\begin{array}{cccccccccccc}
1&0&0&1&0&0&0&0&0&0&0&0\\
0&1&0&0&0&0&1&0&0&0&0&0\\
0&0&1&0&0&0&0&0&0&1&0&0\\
0&0&0&0&1&0&0&1&0&0&0&0\\
0&0&0&0&0&1&0&0&0&0&1&0\\
0&0&0&0&0&0&0&0&1&0&0&1\\
\hline
1&-1&0&0&1&0&0&0&0&0&0&0\\
-1&1&0&0&-1&0&0&0&0&0&0&0\\
0&1&-1&0&0&0&0&0&1&0&0&0\\
0&-1&1&0&0&0&0&0&-1&0&0&0\\
1&0&-1&0&0&1&0&0&0&0&0&0\\
0&0&0&0&1&-1&0&0&1&0&0&0\\
0&0&0&0&-1&1&0&0&-1&0&0&0\\
-1&0&1&0&0&-1&0&0&0&0&0&0
\end{array}\right]}_{A'}
\underbrace{\left[\begin{array}{c}
x_{12}\\x_{13}\\x_{14}\\x_{21}\\x_{23}\\x_{24}\\x_{31}\\x_{32}\\x_{34}\\x_{41}\\x_{42}\\x_{43}
\end{array}\right]}_{x}
\begin{array}{c} = \\ \hline \leq \end{array}
\underbrace{\left[\begin{array}{c}
1\\1\\1\\1\\1\\1\\ \hline 1\\0\\1\\0\\1\\1\\0\\0
\end{array}\right]}_{b'}
\begin{array}{c} b_2 \\ \\ b'_4 \end{array}
$$

It can be easily seen that A' contains unitary columns and duplicate rows (after multiplication by -1). If we eliminate these unitary columns and keep one row of each set of duplicate rows then the matrix A'' so-obtained has the form: $\begin{bmatrix} I \\ N \end{bmatrix}$ where I is the identity matrix and N is a matrix having two non-zeros in each column, one being +1 and the other being -1 and, thus, N is a network matrix (namely, the incidence matrix of a directed graph). Since, network matrices are TU and TU matrices are closed under addition of unitary rows/colums, duplication of rows/columns and multiplication of a row/column by -1 (see e.g. Proposition 2.1 in [7]), it is evident that A' is also a TU matrix. Moreover, since A_1 consists of unitary rows and $A_3 = A_4$, we have that $\bar{A}$ is also a TU matrix. By Theorem 2.1, the result follows. The case for $n = 5$ is proved similarly. $\square$

3.2 The GTT(6) case

Lemma 3.2 *Every extreme $GTT(6)$ matrix is half-integral.*

Sketch of Proof: Doing analogous transformations as in the sketch of the proof of Lemma 3.1, it is enough to prove that the polyhedron $P = \{x | Bx \leq b, x \geq 0\}$ is half-integral, where matrix B, vector x and vector b are provided below. Note also that the class of 2-regular matrices is closed under addition of unitary rows/colums, duplication of rows/columns and multiplication of a row/column by -1 (see Lemmas 6 and 7 in [1]).

$$
\underbrace{\begin{bmatrix}
1 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 1
\end{bmatrix}}_{B}
\underbrace{\begin{bmatrix}
x_{12} \\ x_{13} \\ x_{14} \\ x_{15} \\ x_{16} \\ x_{23} \\ x_{24} \\ x_{25} \\ x_{26} \\ x_{34} \\ x_{35} \\ x_{36} \\ x_{45} \\ x_{46} \\ x_{56}
\end{bmatrix}}_{x}
\leq
\underbrace{\begin{bmatrix}
1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1
\end{bmatrix}}_{b}
$$

By Theorem 2.2 and since the right hand-side (vector b) is integral, showing that B is 2-regular is sufficient to conclude that the vertices of P are half-integral.

□

4 Conclusions and Open Problems

As shown, extreme GTT(n)-matrices are associated with the class of k-regular matrices. For $n \leq 6$, it was proven that extreme GTT(n) matrices are the extreme points of polytopes associated with 2-regular matrices. Thus, it could be conjecture that given any n, the extreme GTT(n) matrices are the extreme points of a polytope associated with a k-regular matrix with $k < n$. However, the resolution of this conjecture might be too ambitious since, to the best of our knowledge, there has not yet been found an extreme GTT(7) matrix with elements other than $\{0, \frac{1}{2}, 1\}$. Thus, the resolution of the following conjecture might be considered as a first significant step:

Conjecture 4.1 *Every extreme GTT(7) matrix is half-integral.*

Moreover, according to the partial results presented in [8], the only known extreme GTT(8) matrices which are non half-integral have elements in $\{0, \frac{1}{3}, \frac{2}{3}, 1\}$. Inspired by these results and taking into account that 6-regular matrices characterize polyhedra whose vertices contain multiples of $\frac{1}{6}$, the following conjecture can be made:

Conjecture 4.2 *Every extreme GTT(8) matrix is $\frac{1}{6}$-integral.*

Consequently, we believe that the current approach based on proving k-regularity of matrices may be an alternative, useful and probably powerful approach in order to resolve problems on characterizing the vertices of the GTT(n) polytopes.

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