A general lower bound on the weak Schur number

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Abstract

For integers k, n with $k, n \geq 1$, the *n-color weak Schur number* $WS_k(n)$ is defined as the least integer N, such that for every n-coloring of the integer interval $[1, N]$, there exists a monochromatic solution $x_1, \ldots, x_k, x_{k+1}$ in that interval to the equation $x_1 + x_2 + \ldots + x_k = x_{k+1}$, with $x_i \neq x_j$, when $i \neq j$. We show a relationship between $WS_k(n+1)$ and $WS_k(n)$ and a general lower bound on the $WS_k(n)$ is obtained.

Keywords: Schur numbers, sum-free sets, weak Schur numbers, weakly sum-free sets, n-coloring.

1 Introduction

For integers $a \leq b$, we shall denote $[a, b]$ the *integer interval* consisting of all $t \in N_+ = \{1, 2, \ldots\}$ such that $a \leq t \leq b$. A function

$$\Delta : [1, N] \longrightarrow \{c_1, \ldots, c_n\},$$

where $c_1, \ldots, c_n \in N_+$ represent different colors, is a n-coloring of the interval $[1, N]$.

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Given a n-coloring Δ and the equation $x_1+\ldots+x_k = x_{k+1}$ in $k+1$ variables, then we say that a solution $x_1, \ldots, x_k, x_{k+1}$ to the equation is monochromatic if and only if $\Delta(x_1) = \Delta(x_2) = \ldots = \Delta(x_{k+1})$.

For integers k, n with k, $n \geq 1$, the *n-color weak Schur number* $WS_k(n)$ is defined as the least integer N, such that for every n-coloring of the integer interval $[1, N]$, there exists a monochromatic solution $x_1, \ldots, x_k, x_{k+1}$ in that interval to the equation $x_1 + x_2 + \ldots + x_k = x_{k+1}$, with $x_i \neq x_j$ when $i \neq j$. In [11], the last author proved that $WS_k(n) \leq t-1$, where $t = r(T_k+1, T_k+1, \ldots, T_k+1)$ is the Ramsey number, with $T_k = (1+k)k/2$, therefore the *n-color weak Schur number* exists.

A set A of integers is called *sum-free* if it contains no elements $x_1, x_2, x_3 \in A$ satisfying $x_1 + x_2 = x_3$ where x_1, x_2 need not be distinct.

Schur [13] in 1916 proved that, given a positive integer n, there exists a greatest positive integer $S_2(n) = N$ with the property that the integer interval $[1, N-1]$ can be partitioned into n *sum-free* sets. The numbers $S_2(n)$ are called Schur numbers. The current knowledge on these numbers for $1 \leq n \leq 7$ is given in Table 1.

n	1	2	3	4	5	6	7
$S_2(n)$	2	5	14	45	161	≥ 537	≥ 1681

Table 1
The first few Schur numbers $S_2(n)$.

The exact value of $S_2(4)$ was given by Baumert [1] and recently $S_2(5)$ has been obtained by Heule [9] . Finally, the lower bounds on $S_2(6)$ and $S_2(7)$ were obtained by Fredricksen and Sweet [7] by considering symmetric sum-free partitions.

Many generalizations of Schur numbers have appeared since their introduction. Now, a set A of integers is called *weakly sum-free* if it contains no pairwise distinct elements $x_1, x_2, x_3 \in A$ satisfying $x_1+x_2 = x_3$. We denote by $WS_2(n)$, the greatest integer N, for which the integer interval $[1, N-1]$ can be partitioned into n weakly sum-free sets $\{A_1, A_2, \ldots, A_n\}$. A result similar to that of Schur [13] was shown by Rado [10] : given $n \geq 1$, there is a greatest integer N for which the interval set $[1, N-1]$ admits a partition into n weakly sum-free sets.

The numbers $WS_2(n)$ are called the *weak Schur numbers* for the equation $x_1 + x_2 = x_3$. The known weak Schur numbers are given in Table 2.

The current state of knowledge concerning $WS_2(n)$ is quite confused. The

n	1	2	3	4	5	6	7	8	9
$WS_2(n)$	3	9	24	67	≥ 197	≥ 583	≥ 1741	≥ 5202	≥ 15597

Table 2
The first few weak Schur numbers $WS_2(n)$.

problem seems to have been first considered in [14], which is Walker's solution to Problem E985 proposed a year earlier, in 1951, by Moser. Walker considered the cases $n = 3, 4$ and 5, and claimed the values $WS_2(3) = 24$, $WS_2(4) = 67$ and $WS_2(5) = 197$. Unfortunately, the short account written by Moser on Walker's solution only gives suitable partitions of $[1, 23]$ for $n = 3$, and no details at all for the cases $n = 4$ and 5. Walker's claimed values of $WS_2(3)$ and $WS_2(4)$ were later confirmed by Blanchard, Harary and Reis using computers [2]. The lower bound $WS_2(5) \geq 197$ has been confirmed in [4]. Whether equality holds in still an open problem. A lower bound on $WS_2(6)$ was obtained by Eliahou et al. [4] and later improved to $WS_2(6) \geq 583$ in [5].The lower bounds for $7 \leq n \leq 9$ were obtained [6] in 2015.

In terms of coloring the $WS_k(n)$ is the least positive integer N such that for every n-coloring of $[1, N]$,

$$\Delta : [1, N] \longrightarrow \{c_1, \ldots, c_n\},$$

where $c_1, \ldots, c_n$ represent n different colors, there exists a monochromatic solution to the equation $x_1 + \ldots + x_k = x_{k+1}$, such that $\Delta(x_1) = \ldots = \Delta(x_k) = \Delta(x_{k+1})$ where $x_i \neq x_j$ when $i \neq j$.

In addition, for 2-coloring, the known weak Schur numbers $WS_k(2)$ are in Table 3.

k	2	3	4	5
$WS_k(2)$	9	24	52	101

Table 3
The first few weak Schur numbers $WS_k(2)$.

The exact values of $WS_k(2)$ for $k = 3, 4$ and the lower bounds were obtained in [11] and $WS_5(2)$ [3] in 2017.

2 A lower bound on $WS_k(2)$ and $WS_k(3)$

Lemma 2.1 *We have* $WS_k(2) \geq \frac{1}{2}(k^3 + 3k^2 - 2k)$ *for any integer* $k \geq 6$.

Proof.

Let Δ be a 2-coloring:

$$\Delta : [1, \frac{1}{2}(k^3 + 4k^2 - 5k + 2)] \longrightarrow \{c_1, c_2\},$$

where c_1, c_2 represent 2 different colors. Let $A_i = \Delta^{-1}(c_i)$ for $i = 1, 2$ thus $[1, \frac{1}{2}(\frac{1}{2}(k^3 + 3k^2 - 2k) + 2)] = A_1 \sqcup A_2$.

We consider the partition of the interval $[1, \frac{1}{2}(k^3 + 3k^2 - 2k)]$:

$$\begin{cases} A_1 = [1, \frac{1}{2}(k^2 + k - 2)] \cup [\frac{1}{2}(k^3 + 2k^2 - k), \frac{1}{2}(k^3 + 3k^2 - 2k - 2)], \\ A_2 = [\frac{1}{2}(k^2 + k), \frac{1}{2}(k^3 + 2k^2 - k - 2)]. \end{cases}$$

We prove that for every i, $1 \le i \le k$, if $x_1, \ldots, x_k \in A_i$ with $x_i \neq x_j$, when $i \neq j$, then $x_1 + \ldots + x_k \notin A_i$.

□

Lemma 2.2 *We have* $WS_k(3) \geq \frac{1}{2}(k^4 + 4k^3 + 2k^2 - 7k + 2)$ *for any integer* $k \geq 2$.

Proof. We defined a 3-coloring,

$$\Delta : [1, \frac{1}{2}(k^4 + 4k^3 + 2k^2 - 7k)] \longrightarrow \{c_1, c_2, c_3\},$$

where c_1, c_2, c_3 represent 3 different colors. Let $A_i = \Delta^{-1}(c_i)$ for $i = 1, 2, 3$.

The interval $[1, \frac{1}{2}(k^4 + 4k^3 + 2k^2 - 7k)] = A_1 \sqcup A_2 \sqcup A_3$ with
$A_1 = A_{1,1} \sqcup A_{1,2} \sqcup A_{1,3}$, $A_2 = A_{2,1} \sqcup A_{2,2}$ and A_3.

The sets $\{A_{1,1}, A_{2,1}\}$ are *weakly k-sum-free*, for 2-coloring in $[1, \frac{1}{2}(k^3+3k^2-2k-2)]$:

$A_{1,1} = [1, \frac{1}{2}(k^2 + k - 2)] \sqcup [\frac{1}{2}(k^3 + 2k^2 - k), \frac{1}{2}(k^3 + 3k^2 - 2k - 2)]$ and
$A_{2,1} = [\frac{1}{2}(k^2 + k)] \sqcup [\frac{1}{2}(k^3 + 2k^2 - k - 2)]$.

The intervals: $A_3 = [\frac{1}{2}(k^3 + 3k^2 - 2k), \frac{1}{2}(k^4 + 3k^3 - k^2 - k - 2)]$,
$A_{1,2} = [\frac{1}{2}(k^4 + 3k^3 - k^2 - k), \frac{1}{2}(k^4 + 3k^3 - 2k - 2)]$,
$A_{2,2} = [\frac{1}{2}(k^4 + 3k^3 - 2k), \frac{1}{2}(k^4 + 4k^3 + k^2 - 6k)]$ and
$A_{1,3} = [\frac{1}{2}(k^4 + 4k^3 + k^2 - 6k + 2), \frac{1}{2}(k^4 + 4k^3 + 2k^2 - 7k)]$.

The sets $\{A_1, A_2, A_3\}$ are *weakly k-sum-free* for the $3 - coloring$ above.

□

3 A general lower bound on $WS_k(n)$

We prove the following relationship between the weak Schur numbers for n-coloring and $(n+1)$-coloring for the equation $x_1 + x_2 + \ldots + x_k = x_{k+1}$, with $x_i \neq x_j$, when $i \neq j$.

Lemma 3.1 *We have $WS_k(n+1) \geq kWS_k(n) + (k-1)p(k)$ for any integer $k \geq 2$, $n \geq 2$, with $p(k) = \frac{1}{2}(k^2 + 5k - 2)$.*

Proof.

We consider a partition with n weakly k-sum-free sets on the interval $[1, WS_k(n) - 1] = A_1 \sqcup \ldots \sqcup A_n$.

Applying the procedure of the Lemma 2.2 we can extend this partition to get an $n+1$- partition weakly k- sum-free. This partition is $\{B_1, B_2, A_3, \ldots, A_n, A_{n+1}\}$, an the intervals are:

$A_{n+1} = [WS_k(n), kWS_k(n) + \frac{1}{2}(k^2 - k - 2)]$,

$A_{1,1} = [kWS_k(n) + \frac{1}{2}(k^2 - k), kWS_k(n) + \frac{1}{2}(2k^2 - 2k - 2)]$,

$A_{2,1} = [kWS_k(n) + \frac{1}{2}(2k^2 - 2k), kWS_k(n) + \frac{1}{2}(k^3 + 3k^2 - 6k)]$,

$A_{1,2} = [kWS_k(n) + \frac{1}{2}(k^3 + 3k^2 - 6k + 2), kWS_k(n) + \frac{1}{2}(k^3 + 4k^2 - 7k)]$,

$B_1 = A_1 \sqcup A_{1,1} \sqcup A_{1,2}$ and $B_2 = A_2 \sqcup A_{2,1}$.

Then $WS_k(n+1) \geq kWS_k(n) + \frac{1}{2}(k^3 + 4k^2 - 7k) + 1 = kWS_k(n) + (k-1)p(k)$.

□

We obtain a general lower bound for $WS_k(n)$ in the following result:

Theorem 3.2 *For the equation $x_1 + \ldots + x_k = x_{k+1}$ such that $x_i \neq x_j$ when $i \neq j$, we have $WS_k(n) \geq q(k)k^{n-2} - p(k)$, with $q(k) = \frac{1}{2}k^3 + 2k^2 + \frac{3}{2}k - 1$ and $p(k) = \frac{1}{2}(k^2 + 5k - 2)$ for $n \geq 2$, $k \geq 2$.*

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