Stabbing convex subdivisions with k-flats

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Abstract

We prove that for every convex subdivision of $\mathbb{R}^d$ into n cells there exists a k-flat stabbing $\Omega\big((\log n/\log\log n)^{1/(d-k)}\big)$ of them. As a corollary we deduce that every d-polytope with n vertices has a k-shadow with $\Omega\big((\log n/\log\log n)^{1/(d-k)}\big)$ vertices.

Keywords: Convex subdivision, stabbing number, Moser's shadow problem

1 Introduction

In a famous list of open problems in combinatorial geometry from 1966 (see [2] and [4]), Moser asked for the largest $\mathfrak{s}(n)$ such that every 3-polytope with n vertices has a 2-dimensional projection with at least $\mathfrak{s}(n)$ vertices. The solution to this problem, popularly known as *Moser's shadow problem*, was implicit in the work of Chazelle, Edelsbrunner and Guibas in 1989 [1] but went unnoticed until recently [3].

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The results of [1] imply that $\mathfrak{s}(n) = \theta(\log n / \log\log n)$. The lower bound was derived from a related result concerning stabbing numbers of convex subdivisions of the plane by lines, that Tóth generalized to stabbing numbers of convex subdivisions of $\mathbb{R}^d$ by lines [5]. He showed that for each convex subdivision of $\mathbb{R}^d$ into n regions there is a line stabbing $\Omega\left((\log n/\log\log n)^{1/(d-1)}\right)$ cells. In the polytopal set-up, this implies that every d-polytope with n-vertices has a 2-dimensional shadow with at least $\Omega\left((\log n/\log\log n)^{1/(d-2)}\right)$ vertices, and the case $d = 2$ for subdivisions and $d = 3$ for convex polytopes is the aforementioned work [1].

In this note, we consider the same questions with lines and 2-dimensional shadows replaced by k-flats and k-shadows (open problems formulated in [5] and [3], respectively). In particular, we prove that for every convex subdivision of $\mathbb{R}^d$ into n cells there exists a k-flat stabbing $\Omega\big((\log n/\log\log n)^{1/(d-k)}\big)$ of its cells, and that every d-polytope with n vertices has a k-shadow with $\Omega\big((\log n/\log\log n)^{1/(d-k)}\big)$ vertices.

2 Stabbing convex subdivisions with k-flats

A *convex subdivision* of $\mathbb{R}^d$ is a finite collection of closed convex d-dimensional sets, the *cells*, that cover $\mathbb{R}^d$ and whose interiors are pairwise disjoint. We say that a *k-flat*, an affine subspace of dimension k, *stabs* a cell if it intersects its interior. Following Chazelle-Edelsbrunner-Guibas [1] and Tóth [5] we prove the following:

Theorem 2.1 *There exists a constant $c_d > 0$ such that for any subdivision of $\mathbb{R}^d$ into n convex cells, and $0 < k < d$, there exists a k-flat stabbing at least $c_d\big(\frac{\log n}{\log\log n}\big)^{\frac{1}{d-k}}$ cells of the subdivision.*

It follows easily from the following lemma.

Lemma 2.2 *Let $\mathcal{S}$ be convex subdivision of $\mathbb{R}^d$ into $n > \ell^{(2\ell)^{d-k}}$ cells, where $0 < k < d$ and $\ell \geq 2$. Then for any generic direction v, either there is a line parallel to v stabbing ℓ cells, or there is hyperplane transversal to v stabbing at least $\ell^{(2\ell)^{d-k-1}}$ cells.*

Proof. Set $\lambda = \ell^{(2\ell)^{d-k-1}}$ and assume that there is no hyperplane transversal to v stabbing at least λ cells. In ℓ steps, we will construct a collection of cells $C_0, \ldots, C_\ell$ that are all simultaneously stabbed by the same line in direction v.

First consider the infinite cells A_0 and B_0 such that $tv \in A_0$ and $-tv \in B_0$ for all $t > 0$ sufficiently large. We set $Z_0 := \mathbb{R}^d \setminus (A_0 \cup B_0)$ and $\mathcal{S}_0 := \mathcal{S} \setminus \{A_0, B_0\}$. Note that all the points $p \in Z_0$ are between A_0 and B_0 in the direction v; that is, the intersection of any line parallel to v with Z_0 is a segment starting at A_0 and finishing at B_0 (and thus if the line intersects the interior of Z_0 it will stab both A_0 and B_0). We call such a set a *v-cylinder*, and A_0 and B_0 its bases. Note also that $\mathcal{S}_0 = \{C \in \mathcal{S} \;:\; Z_0 \cap C \neq \emptyset\}$.

In the following we consider only hyperplanes that are transversal to the direction v. For a given such hyperplane H, we will consider the cells that intersect it in a $(d-1)$-face. We will say that such a cell intersects H *above* or *below* if it intersects H^+ or H^-, respectively, where $H^\pm = \{x \pm tv \;:\; x \in H,\ t > 0\}$. Note that cells that intersect H and are not tangent to H will intersect both above and below.

To find the cells C_i, we will construct a nested family of v-cylinders $Z_\ell \subset Z_{\ell-1} \subset \cdots \subset Z_0$. The bases of Z_i will be A_i and B_i, one of which will be chosen to be C_i. To this end, assume that at the i-th step we have chosen A_i and B_i, and a v-cylinder Z_i that has them as bases. We put $\mathcal{S}_i := \{C \in \mathcal{S} : Z_i \cap C \neq \emptyset\}$. Then we choose a hyperplane H_i that weakly separates A_i from B_i (which exists since A_i and B_i are convex and with disjoint interiors).

Let $\mathcal{T}_i^+$ and $\mathcal{T}_i^-$ be the cells of $\mathcal{S}_i$ that intersect H_i above and below, respectively. Note that by perturbing H_i to $H_i \pm \epsilon v$ with ϵ arbitrarily small, the hyperplane stabs all the cells in $\mathcal{T}_i^\pm$, and hence by assumption $|\mathcal{T}_i^\pm| < \lambda$. Now, we subdivide $H_i^+ \cap Z_i$ into $|\mathcal{T}_i^+|$ v-monotone regions as follows. For each cell $C \in \mathcal{T}_i^+$, we consider the set of points $p \in H_i^+ \cap Z_i$ such that the last region intersected by a ray cast from p in direction $-v$ before hitting H_i is C; that is, that the ray pierces $C \cap H_i$. If H_i is generic, these are just the intersections of $H_i^+ \cap Z_i$ with the cylinders in direction v spanned by each of the regions of $H_i \cap \mathcal{S}_i$. But since the subdivision was not required to be face-to-face and H_i might be tangent to a cell of $\mathcal{S}_i$, $H_i \cap \mathcal{S}_i$ is not necessarily well defined, and we have to distinguish the induced subdivision from above and from below.

The regions constructed this way cover $H_i^+ \cap Z_i$. Subtracting C from its associated region we obtain a v-cylinder Z_C^+ between C and A_i. (Some of these cylinders might be empty, if A_i intersects some cell in $\mathcal{T}_i^+$.) For each $C \in \mathcal{T}_i^-$, we construct analogously a v-cylinder $Z_C^- \subset H_i^- \cap Z_i$ with bases C and B_i.

Note that the union of all these v-cylinders covers all $Z_i \smallsetminus (\mathcal{T}_i^+ \cup \mathcal{T}_i^-)$ (and maybe more, because the cells in $\mathcal{T}_i^+ \cup \mathcal{T}_i^-$ might intersect the cylinders defined by other cells). Hence, by the pigeonhole principle, there must be one of them, that we define to be Z_{i+1}, that intersects at least

$$\frac{|\mathcal{S}_i| - |\mathcal{T}_i^+ \cup \mathcal{T}_i^-|}{|\mathcal{T}_i^+| + |\mathcal{T}_i^-|}$$

cells of $\mathcal{S}_i$. If it belongs to H_i^+ we define $C_i := B_i$, $A_{i+1} := A_i$ and B_{i+1} to be the corresponding cell of $\mathcal{T}_i^+$; and if the cylinder belongs to H_i^- we set $C_i := A_i$, $B_{i+1} := B_i$, and A_{i+1} to be the corresponding cell of $\mathcal{T}_i^-$. We put $\mathcal{S}_{i+1} := \{C \in \mathcal{S} \ : \ Z_{i+1} \cap C \neq \emptyset\}$ and continue inductively until this set is empty.

Now, by induction on i we see that $|\mathcal{S}_i| > \frac{n}{(2\lambda)^i} - 2$ for $0 \leq i < \ell$. Indeed, $|\mathcal{S}_0| = n - 2$ by construction and for $i > 0$ we have

$$|\mathcal{S}_i| > \frac{|\mathcal{S}_{i-1}| - 2\lambda}{2\lambda} > \frac{\frac{n}{(2\lambda)^{i-1}} - 2}{2\lambda} - 1 > \frac{n}{(2\lambda)^i} - 2.$$

In particular,

$$|\mathcal{S}_i| > \frac{\ell^{(2\ell)^{d-k}}}{2^i \ell^{i(2\ell)^{d-k-1}}} - 2 = \frac{\ell^{(2\ell-i)(2\ell)^{d-k-1}}}{2^i} - 2;$$

which is larger than 0 whenever $i < \ell$. Hence, the process does not finish in less than ℓ steps.

Observe that our construction ensures that $\{A_i, B_i\} \neq \{A_{i+1}, B_{i+1}\}$, and that the chosen cells $C_i \in \{A_i, B_i\} \setminus \{A_{i+1}, B_{i+1}\}$ are all distinct. Since the cylinders are nested, $Z_{i+1} \subset Z_i$, any line in direction v through the interior the last non-empty cylinder stabs each of the cells C_i. $\square$

Proof of Theorem 2.1 We prove, by induction on d, that for any subdivision of $\mathbb{R}^d$ into $n > \ell^{(2\ell)^{d-k}}$ convex cells, there exists a k-flat intersecting at least ℓ cells. For $d = r$ there is nothing to prove. If $d > r$, assume that for any convex subdivision of $\mathbb{R}^{d-1}$ into $\ell^{(2\ell)^{d-1-r}}$ cells there exists a k-flat intersecting at least ℓ-cells.

Applying the lemma to the original subdivision we get an alternative: if the subdivision contains a line stabbing ℓ elements then we are done as any k-flat containing the line intersects at least this number of cells. Otherwise there exists a hyperplane H intersecting $\ell^{(2\ell)^{d-r-1}}$ cells. By the induction hypothesis on the subdivision induced on H, there is a k-flat contained in H that intersects at least ℓ cells. $\square$

3 Moser's shadow problem in high dimensions

A *k-shadow* of a convex polytope P is a k-dimensional polytope that is the image of P under a linear map.

Theorem 3.1 *There exists a constant $c'_d > 0$ such that for $1 < k < d$, every d-polytope with n vertices has a k-shadow with at least $c'_d\left(\frac{\log n}{\log\log n}\right)^{\frac{1}{d-k}}$ vertices.*

Proof. Let P be a d-polytope with n vertices, and π the orthogonal projection of $\mathbb{R}^d$ onto a k-dimensional linear subspace L. Then, if $\mathcal{N}(P)$ is the normal fan of P, the normal fan of $\pi(P)$ is isomorphic to $\mathcal{N}(P) \cap L$ (see [6, Lem. 7.11]).

Now consider the spherical subdivision of $\mathbb{S}^{d-1}$ induced by $\mathcal{N}(P)$. It has n $(d-1)$-dimensional regions, one for each vertex, and after a suitable rotation we might assume that the lower hemisphere intersects at least $\frac{n}{2}$ of them. Central projection from the origin maps this subdivision of the lower hemisphere onto a convex subdivision of $\mathbb{R}^{d-1}$ into at least $\frac{n}{2}$ cells. By Theorem 2.1, there is a $(k-1)$-flat that intersects at least

$$c_d\left(\frac{\log\frac{n}{2}}{\log\log\frac{n}{2}}\right)^{\frac{1}{(d-1)-(k-1)}} \geq c'_d\left(\frac{\log n}{\log\log n}\right)^{\frac{1}{d-k}}$$

cells of this subdivision. The pre-image of this $(k-1)$-flat spans a k-dimensional linear subspace L intersecting at least $c'_d(\frac{\log n}{\log\log n})^{\frac{1}{d-k}}$ cells of $\mathcal{N}(P)$. Therefore, the orthogonal projection of P onto L is a k-shadow with at least this many vertices. □

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