The complete classification of empty lattice 4-simplices

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Abstract

In previous work we classified all empty 4-simplices of width at least three. We here classify those of width two. There are 2 two-parameter families that project to the second dilation of a unimodular triangle, 29 + 23 one-parameter families of them that project to hollow 3-polytopes, and 2282 individual ones that do not.

Keywords: lattice polytopes, empty simplices, classification of empty polytopes.

1 Introduction

Recall that a *lattice polytope* is the convex hull in $\mathbb{R}^d$ of finitely many points of $\mathbb{Z}^d$ (or of any other geometric lattice $\Lambda \subset \mathbb{R}^d$). A *lattice d-simplex* is a lattice d-polytope whose vertices are affinely independent and with no other lattice point. Equivalently, it is a lattice d-polytope with exactly $d+1$ lattice points. By *classification* we mean modulo *unimodular equivalence*.

Lattice polytopes have been widely studied for their relations to algebraic geometry and integer optimization, among other fields. For example, empty

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lattice simplices correspond to the so called *terminal quotient singularities* in the minimal model program of Mori for the birational classification of algebraic varieties. Their classification in dimension three is sometimes dubbed the *terminal lemma*, and quite some effort has been devoted towards the classification of 4-dimensional ones. In particular, Mori et al. [7] conjectured a classification of the empty 4-dimensional simplices of *prime volume* into a three-parameter family, one two-parameter family, 29 one-parameter families, and a finite list of exceptions, of volumes up to 419. This classification was later proved by Bober [3] (see also [4,9]).

For the non-prime case, [1] claimed that the classification of Mori et al. extended without change (except for an increase in the number of exceptional simplices) but this statement was found in [2] to be false, where additional infinite families were found. In our previous work [6] we completely classified empty 4-simplices of width three or more, proving that the maximum volume among them is 179, as conjectured in [5]. We here report on a complete classification of simplices of width two, which finishes the task since simplices of width one are easy to classify in arbitrary dimension.

2 Preliminaries on the classification

2.1 Normalized volume and lattice width

We first introduce the two basic invariants for the classification of empty simplices (or more general lattice polytopes), already mentioned in the previous paragraphs: their *normalized volume* and their *width*. Let Λ be a geometric d-dimensional lattice in $\mathbb{R}^d$.

A *unimodular simplex* is a lattice simplex whose vertices form an affine basis for Λ. All unimodular simplices are unimodularly equivalent and, in particular, they have the same volume, sometimes called the *determinant* of Λ. The *normalized volume* in $\mathbb{R}^d$ is the Euclidean volume normalized to the lattice, so that unimodular simplices have volume one. For example, if $\Lambda = \mathbb{Z}^d$ then the normalized volume equals the Euclidean volume multiplied by $d!$. In particular, the normalized volume of every lattice polytope is a positive integer, and it equals one exactly for unimodular simplices.

The *width* of a body $K \subset \mathbb{R}^d$ with respect to a linear functional $f : \mathbb{R}^d \to \mathbb{R}$ is the length of the interval $f(K)$, that is, the difference $\max_{x\in K} f(x) - \min_{x\in K} f(x)$. We call *(lattice) width* of a lattice polytope P the minimum width of P with respect to all non-constant affine functionals with $f(\Lambda) \subset \mathbb{Z}$.

The classification of empty simplices of dimension up to three is classical.

In dimension two every empty triangle is *unimodular*. In dimension three there are infinitely many empty simplices which were classified by White [10] giving a complete characterization. All of them have width one.

In dimension 4 the same is not true. Haase and Ziegler in [5] showed that there are infinitely many non-equivalent lattice 4-simplices of width two and conjectured both parts of the following recent result:

Theorem 1 (i) *There are only finitely many empty* 4*-simplices of width greater than two [2].*

(ii) *All empty* 4*-simplices of width greater than two have volume between* 41 *and* 179*. There are exactly* 178 *classes of witdth three, a single one of width four, and none of larger width [6].*

Empty simplices of width 1 are easy to classify à la White (see details below), so most of the task in this paper is to classify empty 4-simplices of width two.

2.2 Lattice simplices as finite subgroups of the torus

Before giving details about the classification itself, let us explain an intrinsic way of representing a lattice simplex (empty or not).

Let $P = \mathrm{conv}(v_0, \dots, v_d)$ be a lattice d-simplex of a certain volume V with respect to a given lattice Λ. Let Λ_0 be the affine sublattice generated by $v_0, \dots, v_d$, so that $V = |\Lambda : \Lambda_0|$. Then, every point $p \in \mathrm{aff}(\Lambda)$ can be uniquely reperesented in barycentric coordinates by a vector $b = (b_0, \dots, b_d)$ with $\sum_i b_i = 1$, meaning by this that $p = \sum_i b_i p_i$. Moreover, two points p, p' have barycentric coordinates differing by an integer vector if and only if they lie in the same translate of Λ_0.

In particular, if we let

$$\mathbb{T}^d := \{b \in \mathbb{R}^{d+1} : \sum_i b_i = 1\} / \{b \in \mathbb{Z}^{d+1} : \sum_i b_i = 1\} \cong \mathbb{R}^d/\Lambda_0$$

be the homogeneous real torus of dimension d, then every lattice simplex P of volume V induces a subgroup $G(P) \cong \Lambda/\Lambda_0$ of order V of $\mathbb{T}^d$. Moreover:

Proposition 2 *Two simplices P and P' are equivalent if and only if $G(P)$ and $G(P')$ are the same group modulo permutation of coordinates.*

That is, we can represent a lattice simplex P as a finite subgroup $G(P)$ of $\mathbb{T}^d$.

We can relax the condition that $\sum_i b_i = 1$ for our barycentric coordinates and ask only that $\sum_i b_i \in \mathbb{Z}$ taking into account that

$$\{(b_0,\dots,b_d)\in\mathbb{R}^{d+1}:\sum_i b_i=1\}/\{(b_0,\dots,b_d)\in\mathbb{Z}^{d+1}:\sum_i b_i=1\}$$
$$=\{(b_0,\dots,b_d)\in\mathbb{R}^{d+1}:\sum_i b_i\in\mathbb{Z}\}/\{(b_0,\dots,b_d)\in\mathbb{Z}^{d+1}\}.$$

Thus, every element of $\mathbb{T}^d$ can be uniquely represented by a vector in $[0,1)^{d+1}$ with integer sum of coordinates. P is empty if, and only if, no element of $G(P)$ (expressed in this canonical form) has sum of coordinates equal to 1.

We call a lattice simplex P *cyclic* if $G(P)$ is a cyclic group. Barile et al. [1, Theorem 1] showed that every empty lattice simplex of dimension four is cyclic, so that we can identify by a generator of $G(P)$. Thus:

Corollary 3 *Every cyclic d-simplex (in particular, every empty d-simplex with $d\le 4$) of volume V can be described by a $(d+1)$-tuple of numbers in $\frac{1}{V}\mathbb{Z}^{d+1}$. Two tuples represent the same simplex if (and only if) they generate the same (modulo permutation of coordinates) subgroup of $\mathbb{T}^{d+1}$.*

Example 4 *The empty 3-simplex $T(p,q)=\operatorname{conv}\{0,(1,0,0),(0,0,1),(p,q,1)\}$ has the associated 4-tuple $\frac{1}{q}(-p,p,-1,1)$, since $\Lambda=\mathbb{Z}^3$, $\Lambda_0=\{(a,b,c)\in\mathbb{Z}^3 : b=0 \pmod q\}$ and Λ/Λ_0 is generated by*
$$(0,1,0)=\left(1-\frac{p}{q}\right)(0,0,0)+\frac{p}{q}(1,0,0)-\frac{1}{q}(0,0,1)+\frac{1}{q}(p,q,1).$$

2.3 Hollow projections of hollow polytopes

One last ingredient that is useful in order to state (and prove) our classification is the following result of Nill and Ziegler. A *hollow* polytope is a lattice polytope with no interior lattice points. We say that a hollow polytope $P\subset\mathbb{R}^d$ projects to a hollow polytope $Q\subset\mathbb{R}^{d'}$, $d'<d$, if there is an affine integer map $\pi:\mathbb{R}^d\to\mathbb{R}^{d'}$ with $\pi(\mathbb{Z}^d)=\mathbb{Z}^{d'}$ and $\pi(P)=Q$.

Theorem 5 ([8]) *For each dimension d there is only finitely many hollow d-polytopes that do not project to any hollow $(d-1)$-polytope.*

In particular, an interesting invariant of hollow polytopes (in particular, empty simplices) is what is the minimal dimension of a hollow polytope they project to. Observe that a hollow polytope projects to dimension 1 if and only if it has width one.

3 The classification

Theorem 6 (Classification of empty 4-simplices) *Let P be an empty 4-simplex of normalized volume V and let $d'\in\{1,2,3,4\}$ be the minimum di-*

$\frac{1}{2}(0,0,1,1,0) + \frac{1}{V}(6,-2,-12,4,4)$	$\frac{1}{3}(0,0,2,1,0) + \frac{1}{V}(-9,6,3,3,-3)$
$\frac{1}{2}(1,0,0,0,1) + \frac{1}{V}(8,-6,2,-8,4)$	$\frac{1}{3}(1,0,2,0,0) + \frac{1}{V}(9,-9,3,-6,3)$
$\frac{1}{2}(0,0,1,0,1) + \frac{1}{V}(8,-4,-12,6,2)$	$\frac{1}{3}(0,0,1,2,0) + \frac{1}{V}(-9,3,6,6,-6)$
$\frac{1}{2}(1,0,0,0,1) + \frac{1}{V}(4,6,-2,-16,8)$	$\frac{1}{3}(0,0,1,2,0) + \frac{1}{V}(12,-6,-12,3,3)$
$\frac{1}{2}(0,1,1,0,0) + \frac{1}{V}(2,-12,4,12,-6)$	$\frac{1}{3}(1,0,2,0,0) + \frac{1}{V}(9,-18,6,6,-3)$
$\frac{1}{2}(1,0,1,0,0) + \frac{1}{V}(12,-16,8,-6,2)$	$\frac{1}{3}(1,0,2,0,0) + \frac{1}{V}(12,-18,3,6,-3)$
$\frac{1}{2}(0,1,0,0,1) + \frac{1}{V}(2,12,-8,-12,6)$	$\frac{1}{3}(1,0,2,0,0) + \frac{1}{V}(12,-9,3,-12,6)$
$\frac{1}{2}(1,0,0,0,1) + \frac{1}{V}(8,6,-2,-24,12)$	$\frac{1}{3}(1,0,2,0,0) + \frac{1}{V}(6,-3,6,-18,9)$
$\frac{1}{2}(0,1,0,0,1) + \frac{1}{V}(6,-2,8,-24,12)$	$\frac{1}{3}(0,0,1,1,1) + \frac{1}{V}(3,-18,6,18,-9)$
$\frac{1}{4}(2,1,1,0,0) + \frac{1}{V}(12,-12,4,-8,4)$	$\frac{1}{6}(1,0,0,4,1) + \frac{1}{V}(6,-18,6,12,-6)$
$\frac{1}{4}(0,1,1,0,2) + \frac{1}{V}(4,8,-4,-16,8)$	
$\frac{1}{4}(0,0,1,2,1) + \frac{1}{V}(4,-16,4,16,-8)$	
$\frac{1}{4}(0,1,1,0,2) + \frac{1}{V}(4,12,-4,-24,12)$	

Table 1
The 23 non-primitive quintuples.

mension of a hollow polytope that P projects to. Then P lies in one of the following explicitly described categories, depending on d':

$\boxed{d' = 1:}$ *(That is, P has width one). Then P can be represented by a 5-tuple of the form $\frac{1}{V}(-a,-b,a+b,-1,1)$, where $\gcd(a,b,V)=1$. P is equivalent to the simplex*

$$\mathrm{conv}\{(0,0,0,0),(1,0,0,0),(0,1,0,0),(0,0,0,1),(a,b,V,1)\}.$$

$\boxed{d' = 2:}$ *P projects to the doubled unimodular triangle, the only hollow 2-polytope of width >1. There are two possibilities for the tuple, namely:*

$$\frac{1}{V}(a,-2a,b,-2b,a+b), \qquad \textit{and}$$
$$\frac{1}{2}(0,0,1,0,1)+\frac{1}{V}(2,-1,-1,a,-a).$$

$\boxed{d' = 3:}$ *There are 29 + 23 possibilities for the tuple. The first 29 are the "stable quintuples" identified by Mori, Morrison and Morrison [7]. The other 23 are "non-primitive families" that we list in Table 1, separated according to their index, which is an integer in $\{2,3,4,6\}$.*

$\boxed{d' = 4:}$ *There are exactly 2461 (classes of) empty* 4*-simplices that do not project to a hollow* 3*-polytope. Their volumes range from 24 to 419 and according to their width they fall in 2282, 178 and 1 classes of widths 2, 3 and 4, respectively.*

Observe that not all possibilities for the parameters a, b, and V in the statement produce empty simplices. Our claim is that all empty 4-simplices fall into this classification. (It is not difficult, but tedious, to specify the exact conditions on a, b and V that make the simplices empty, for each case).

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