Variance of the sum of independent random variables in spheres ⋆

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Abstract

The sum of random variables (errors) is the key element both for its statistical study and for the estimation and control of errors in many scientific and technical applications. In this paper we analyze the sum of independent random variables (independent errors) in spheres. This type of errors are very important, for example, in quantum computing. We prove that, given two independent isotropic random variables in an sphere, X_1 and X_2, the variance verifies $V(X_1 + X_2) = V(X_1) + V(X_2) - \frac{V(X_1)V(X_2)}{2}$ and we conjecture that this formula is also true for non-isotropic random variables.

Keywords: Independent random variables on spheres, variance, independent errors on spheres, quantum computing errors.

⋆ This work is an extended abstract containing only the statement of our results. The full article, with proofs, is under consideration in the *Journal of the EMS.*

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1 Introduction

Given an error (random variable) X in $\mathbb{R}^d$, with mean μ, the most useful parameter to quantify the error is the variance of X, $V(X) = E[\|X - \mu\|^2]$. Let's assume that instead of an error X we have many, $X_1,\dots X_k$, and that they add up as, for example, in stochastic processes that change in discrete time. It is well known that if they are independent the variance verifies

$$V(X_1 + \cdots + X_k) = V(X_1) + \cdots + V(X_k)$$

This statement is called the Bienaymé formula ([1],1853). Nevertheless, nothing is practically known if the errors are defined in the d dimensional sphere of radius 1 centered in 0, S^d; surely because this analysis is more complicated and there have not been important applications so far that necessarily have to be studied from this point of view.

This approach fits, for example, the study of errors in the position of a robotic arm whose components are connected by joints allowing rotational motion. However, the sensors that detect the position of the robotic arm allow a very effective control of the error in its position, making a deep analysis of these errors unnecessary.

The behavior of errors in quantum computing [3,4] also requires the use of random variables in spheres and, in this case, the laws of quantum mechanics do not allow any self-correcting system like that of sensors in robotic environments. Therefore the development of the theory of random variables in spheres has become essential to address the challenge of quantum computing and this work means an important achievement.

The outline of the article is as follows: In section 2 we set up notations and discuss some basic properties of the introduced model. In section 3 we give a hint of the proof of the formula for the variance of the sum of two independent isotropic random variables in an sphere. In section 4 we illustrate with an example the conjecture that the previous formula is also true for non-isotropic errors. Finally in section 5 we explain the quantum decoherence relating it to the behavior of the variance of independent errors.

2 Notations and basic properties

The $d-$dimensional sphere of radius 1 centered at 0 is the subset of $\mathbb{R}^{d+1}$ $S^d = \{x \in \mathbb{R}^{d+1} \mid \|x\| = 1\}$. Given a point $x = (x_0, \dots, x_d) \in S^d$ its polar coordinates are $(\theta_0, \theta_1, \dots, \theta_{d-1})$ where

$$\begin{array}{ll} 0 \leq \theta_0, \ldots, \theta_{d-2} \leq \pi & \\ & \text{and} \quad x_j = \sin(\theta_0) \cdots \sin(\theta_{j-1}) \cos(\theta_j),\ 0 \leq j \leq d-1 \\ 0 \leq \theta_{d-1} < 2\pi & \\ & x_d = \sin(\theta_0) \cdots \sin(\theta_{d-1}) \end{array}$$

From now on, we will consider only random variables (errors) X on S^d centered on the point $P = (1, 0, \ldots, 0)$ ($\theta_p = (0, \ldots, 0)$ in polar coordinates), i.e. with mean P. This assumption does not imply a loss of generality, because we can always transfer the mean to the point P by a transformation of $O(d+1)$ or $U(d+1)$. We will also always work in polar coordinates and will define the random variables by means of density functions $f(\theta_0, \theta_1, \ldots, \theta_{d-1})$.

Henceforward we will denote random variable by the abbreviation r.v. and random variables by r.vs.

The variance of a r.v. in S^d, X, is the mean of its quadratic deviation, i.e. $E[\|X - P\|^2] = E[\|(x_0 - 1, x_1, \ldots, x_d)\|^2] = E[2 - 2x_0]$. And, in polar coordinates, the variance is $E[2 - 2\cos(\theta_0)]$.

Definition 2.1 The variance of a r.v. in S^d, X, centered on the point P, is $V(X) = E[2 - 2\cos(\theta_0)]$.

Note that the variance will take values between 0, if the density function is a delta at point P, and 4, if the density function is a delta at point $-P$. Therefore, the Bienaymé formula can not hold for r.vs in S^d.

Definition 2.2 A r.v. in S^d is isotropic if its density function depends exclusively on θ_0.

For simplicity, we will assume that the density function of any isotropic r.v. depends on $\cos(\theta_0)$. Indeed $g(\theta_0) = f(\cos(\theta_0))$, where $f = g \circ \arccos$.

The "reasonable" density functions of isotropic r.vs (errors) are non-increasing with respect to the variable θ_0. This means that the error probability (density function) does not grow when the error does. In this case, the variance is upper bounded by 2 and takes the value 2 when the density function is constant in S^d. For example, in quantum computing this means that the information of a quantum algorithm has been completely lost.

In order to analyze the variance of the sum of independent isotropic r.vs we need to obtain the expression of the resulting density function. Let X_1 and X_2 be independent isotropic r.vs with density function f_1 and f_2 respectively and let $X = X_1 + X_2$ and f its density function. The sum of r.vs (errors) occurs as follows: initially there is no error and, therefore, our system is in P; when the first error occurs the system changes to x', with a probability that depends on $\|x' - P\|^2$; and when the second error occurs the system goes to x, with a probability that depends on $\|x' - x\|^2$ (isotropic error with respect

to x'). Then:

$$f(x) = \int_{S^d} f_1 \, f_2 \, dx' \tag{1}$$

Theorem 2.3 *Given two independent isotropic r.vs in S^d, X_1 and X_2, then the sum $X = X_1 + X_2$ is also an isotropic r.v. in S^d.*

Proof. Given x_0' and x_0, let us consider S' and S, the $d-1$ dimensional spheres of radius $1-x_0'^2$ and $1-x_0^2$ centered on $(x_0', 0, \ldots, 0)$ and $(x_0, 0, \ldots, 0)$ respectively. S' and S are "parallels" of S^d passing through x_0' and x_0 respectively, if we consider P as the S^d polar point. By the symmetry of S' and S and the isotropy of f_1 and f_2 we can conclude that the integral (1), restricted to S', is constant in S. Therefore the integral (1) is also constant in S and, consequently, the sum X of the two independent isotropic r.vs is an isotropic r.v. in S^d. □

Corollary 2.4 *Given two independent isotropic r.vs X_1 and X_2 in S^d, $d \geq 3$, then the density function of the sum $X = X_1 + X_2$ is*

$$f(\theta_0) = |S^{d-2}| \int_0^\pi \int_0^\pi f_1(\alpha) \, f_2(\beta) \, \sin^{d-1}(\theta_0') \sin^{d-2}(\theta_1') \, d\theta_0' \, d\theta_1' \tag{2}$$

where $|S^{d-2}|$ is the area of S^{d-2}, f_1 and f_2 the density functions of X_1 and X_2 respectively, $\alpha = \cos(\theta_0')$ and $\beta = \cos(\theta_0)\cos(\theta_0') + \sin(\theta_0)\sin(\theta_0')\cos(\theta_1')$.

In quantum computing [4] algorithms work with $n-$qubits that belong to a complex vector space of dimension 2^n and have norm 1. Therefore quantum computation errors can be considered as r.vs in S^d with $d = 2^{n+1} - 1$ and the results presented in this article can be applied to them.

3 Variance of the sum of independent isotropic random variables

We shall work with the family of functions

$$g_k(\theta) = \frac{1 + \cos^k \theta}{\int_{S^d}(1 + \cos^k \theta) dS^d}, \theta \in [0, \pi], k \in \mathbb{N}$$

The functions g_k are a family of isotropic density functions on S^d (as they are bounded, positive, and their integral is 1). Furthermore, they are a base of the space of isotropic density functions. This fact can be proved using the Stone-Weierstrass Theorem ($\{1, x, x^2, \ldots\}$ is a complete base on $L^2[-1, 1]$), and deducing from it that $\{\cos^k(\theta) | k \geq 0\}$ is a complete base on $L^2[0, \pi]$.

We prove that $V(g_k + g_l) = \frac{2(V(g_k)+V(g_l))-V(g_k)V(g_l)}{2}$ for every k, l. It suffices to compute the integral expressions: for k or l even, both sides of the formula

equal 2. For k and l odd, proving the formula reduces to check that

$$\sum_{c=0}^{b} \binom{b}{c} (2c+d-2)!!(2(a+b-c)+1)!! = \frac{(2a+1)!!(d-2)!!(2(a+b)+d+1)!!}{(2a+d+1)!!},$$

which can be shown through a combinatorial argument. As there are $(2n-1)!!$ increasing ordered trees with $n+1$ vertices ([2]), we count in two different ways the number of increasing ordered rooted trees with basic subtrees of size $\lfloor d/2 \rfloor$ and $a+2$, after adding b vertices with greater labels. The arguments are slightly different for even and odd d.

Theorem 3.1 *Given two independent isotropic r.vs in S^d, X_1 and X_2,*

$$V(X_1+X_2) = V(X_1) + V(X_2) - \frac{V(X_1)V(X_2)}{2}$$

Proof. Let us decompose the density functions of X_1 and X_2, f_1 and f_2 respectively, into infinite linear combinations of density functions of the base $B = \{ g_k \mid k \in \mathbb{N}^+ \}$ of $L^2[0,\pi]$:

$$f_1 = \sum_{j=0}^{\infty} \alpha_j g_j \quad \text{and} \quad f_2 = \sum_{k=0}^{\infty} \beta_k g_k \quad \text{such that} \quad \sum_{j=0}^{\infty} \alpha_j = \sum_{k=0}^{\infty} \beta_k = 1$$

The result is obtained using that the formula holds for the family g_k. □

4 The non-isotropic case

We conjecture that Theorem 3.1 also holds for general (non-isotropic) r.vs on S^d. Next we check it for an isotropic and a non-isotropic r.v.

Example 4.1 Let us consider the density functions on S^7:

$$g_1 = \frac{2+\cos(\theta_0)\sin^2(\theta_1)+\cos(\theta_1)\sin(\theta_2)}{\int_{S^7}(2+\cos(\theta_0)\sin^2(\theta_1)+\cos(\theta_1)\sin(\theta_2))dS^7}, \qquad g_2 = \frac{1+\cos(\theta_0)}{\int_{S^7}(1+\cos(\theta_0))dS^7}.$$

A computation shows that in this case,

$$\frac{2(V(g_1)+V(g_2)) - V(g_1)V(g_2)}{2} = \frac{445}{224} = V(g_1+g_2)$$

(generalizing corollary 2.4 to get the density function of g_1+g_2).

5 Decoherence of quantum states

Let us assume that we are running a fault-tolerant quantum algorithm and that, each time the correction circuit is applied, a decoherence error, that has

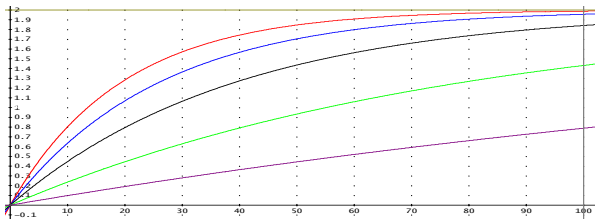
an isotrope component with variance σ, occurs. The behavior of this error can be analyzed through a stochastic process that changes in discrete time. If the complete process involves k errors and these are independent, because they occur at different times, then the final error of the $n-$qubit generated by the algorithm will be:

$$V(E_1 + \cdots + E_k) = 2 - 2\left(1 - \frac{\sigma}{2}\right)^k$$

The above formula can be easily demonstrated by induction from Theorem 3.1, and can be generalized for different variances $\sigma_1, \ldots, \sigma_k$:

$$V(E_1 + \cdots + E_k) = \sum_{j=1}^{k} (-1)^{j+1} \frac{s_j}{2^{j-1}} \quad \text{where} \quad \begin{array}{l} s_1 = \sigma_1 + \sigma_2 + \cdots + \sigma_k \\ s_2 = \sigma_1\sigma_2 + \sigma_1\sigma_3 + \cdots + \sigma_{k-1}\sigma_k \\ \cdots \\ s_k = \sigma_1\sigma_2\cdots\sigma_k \end{array}$$

The figure below shows the error behavior when $k = 100$ for variances $\sigma = 0.1,\ 0.075,\ 0.05,\ 0.025$ and 0.01. Mark the exponential growth towards the asymptotic value $\bar{\sigma} = 2$. Note also that, to keep the error reasonably controlled, the variance of the errors that occur in each time interval must be smaller than $\frac{1}{k}$ (the inverse of the algorithm's execution time).



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