On the chromatic number of a subgraph of the Kneser graph

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Abstract

Let n and k be positive integers with $n \geq 2k$. Consider a circle C with n points 1, $\ldots, n$ in clockwise order. The *interlacing graph* $\mathrm{IG}_{n,k}$ is the graph with vertices corresponding to k-subsets of $[n]$ that do not contain two adjacent points on C, and edges between k-subsets P and Q if they *interlace*: after removing the points in P from C, the points in Q are in different connected components. In this paper we prove that the circular chromatic number of $\mathrm{IG}_{n,k}$ is equal to n/k, hence the chromatic number is $\lceil n/k \rceil$, and that its independence number is $\binom{n-k-1}{k-1}$.

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1 Introduction

Let n and k be positive integers with $n \geq 2k$. The Kneser graph $\mathrm{KG}_{n,k}$, first introduced by Martin Kneser in [4], is the graph with vertex set corresponding to k-subsets of $[n] := \{1, \ldots, n\}$, where two vertices are adjacent if the corresponding sets are disjoint. Kneser conjectured that the chromatic number of $\mathrm{KG}_{n,k}$ is $n - 2k + 2$. In [5], Lovász proved this conjecture using topological methods. The Schrijver graph $\mathrm{SG}_{n,k}$ is the subgraph of $\mathrm{KG}_{n,k}$ induced by those vertices that correspond to k-subsets of $[n]$ not containing adjacent elements in $[n]$ (here 1 and n are adjacent). In [8], Schrijver showed that $\mathrm{SG}_{n,k}$ is a vertex-critical subgraph of $\mathrm{KG}_{n,k}$ and also has chromatic number $n - 2k + 2$. (Vertex-critical means that the deletion of any vertex reduces the chromatic number.) Another famous result regarding the Kneser graph is the Erdős-Ko-Rado theorem [2], which says that the maximum size of an independent set of $\mathrm{KG}_{n,k}$ is $\binom{n-1}{k-1}$.

Let G a finite graph, a *circular coloring* of size n/k is an assignment $\chi : V(G) \to \mathbb{Z}/n\mathbb{Z}$ such that $\chi(v_1) - \chi(v_2) \in \{k, k+1, \ldots, -k\} \mod n$ if $\{v_1, v_2\}$ is an edge of the graph. The *circular chromatic number* of a finite graph G, $\chi_{\mathrm{circ}}(G)$, can be defined as the minimal rational number n/k for which there exists a circular coloring of size n/k. The *circular clique* $K_{n/k}$ is the graph on vertices $\{0, \ldots, n-1\}$ such that two vertices are adjacent if their distance in $\mathbb{Z}/n\mathbb{Z}$ is larger or equal than k. If G is a finite graph then $\lceil \chi_{\mathrm{circ}}(G) \rceil = \chi(G)$ (see, for instance, [10]). In [1], Chen confirmed the conjecture from [3] that $\chi_{\mathrm{circ}}(\mathrm{KG}_{n,k}) = \chi(\mathrm{KG}_{n,k})$ which was previously known for some cases, such as for even n for the Schrijver graph [6,9].

In this paper we consider a subgraph of the Kneser graph. If P and Q are two k-subsets of $[n]$, $P = \{1 \leq p_1 < \ldots < p_k \leq n\}$ and $Q = \{1 \leq q_1 < \ldots < q_k \leq n\}$, then P and Q are *interlacing* if either

$$1 \leq p_1 < q_1 < p_2 < q_2 < \cdots < p_k < q_k \leq n$$

or

$$1 \leq q_1 < p_1 < q_2 < p_2 < \cdots < q_k < p_k \leq n.$$

By distributing the elements of $[n]$ in clockwise order around a circle we may view P and Q as k-polygons with points on the circle. Then P and Q are interlacing if removing the points of P divides the circle into intervals that each contain one point of Q. We use this analogy to refer to k-subsets in $[n]$ as k-polygons, or just polygons when k is understood. We say that a polygon that does not contain two adjacent points on the circle is *admissible*. The *interlacing graph* $\mathrm{IG}_{n,k}$ is the graph whose vertices correspond to admissible

k-polygons on $[n]$, and where two vertices are adjacent if the corresponding polygons are interlacing. As non-admissible polygons would give rise to isolated vertices in the interlacing graph, only admissible polygons are considered. Note that $\mathrm{IG}_{n,k}$ also is a subgraph of the Schrijver graph $\mathrm{SG}_{n,k}$.

1.1 Main results

Our main result is the following.

Theorem 1.1 *The circular chromatic number of* $\mathrm{IG}_{n,k}$ *is equal to* n/k.

Thus, we obtain that:

Corollary 1.2 *The chromatic number of* $\mathrm{IG}_{n,k}$ *is equal to* $\lceil n/k \rceil$.

We also determine the independence number of the interlacing graph.

Proposition 1.3 *The independence number of* $\mathrm{IG}_{n,k}$ *is* $\binom{n-k-1}{k-1}$.

To prove Theorem 1.1 we find a circular clique $K_{n/k}$ as a subgraph in $\mathrm{IG}_{n,k}$. Afterwards we find a circular coloring χ of size n/k, or a graph homomorphism from $\mathrm{IG}_{n,k}$ to $K_{n/k}$. The vertex color classes induced by the coloring χ can be naturally grouped into stable sets of maximum (or almost maximum) size.

The interlacing graph has connections with triangulations of cyclic polytopes. Note that if $k = 2$, two non-interlacing polygons on $[n]$ are just two non-crossing lines between vertices of an n-polygon. A maximal set of pairwise non-interlacing polygons is a triangulation. In [7], Oppermann and Thomas generalized this observation to higher dimensions: the set of triangulations of the cyclic polytope with n vertices in dimension $2k-2$, are in bijection with the independent sets of admissible polygons in $\mathrm{IG}_{n,k}$ of maximal size. (The cyclic polytope $C(n, 2k-2)$ is the convex hull of n distinct points in $\mathbb{R}^{2k-2}$ that are obtained as evaluations of the curve defined by $P(x) = (x, x^2, \ldots, x^{2k-2})$, which is called the moment curve.) In particular, the chromatic number of $\mathrm{IG}_{n,k}$ gives the minimal size of a partition of the $(k-1)$-dimensional internal simplices of $C(n, 2k-2)$ (see [7]) in which no two simplices in each part internally intersect. The proof of Proposition 1.3 exploits this connection and follows the framework developed in [7].

2 Graph parameters of $\mathrm{IG}_{n,k}$

2.1 The independence number

The proof of Proposition 1.3 uses a standard counting argument.

Lemma 2.1 *The number of admissible k-polygons on $[n]$ containing a specific point on the circle is $\binom{n-k-1}{k-1}$. In particular, the number of vertices of $\mathrm{IG}_{n,k}$ is $\frac{n}{k}\binom{n-k-1}{k-1}$.*

The combination of Lemma 2.1 with the techniques and arguments appearing in [7] show Proposition 1.3

2.2 Lower bound for the circular chromatic number

The following lemma gives the lower bound to the circular chromatic number for the interlacing graph.

Lemma 2.2 *Let n, k be positive integers, $n \geq 2k$, and let n', k' be coprime positive integers such that $n'/k' = n/k$. The subgraph of $\mathrm{IG}_{n,k}$ induced by the n' polygons $\{P^j\}_{j\in[0,n-1]}$,*

$$P^j = \{j + n, j + \lceil n/k\rceil, j + \lceil 2n/k\rceil, \ldots, j + \lceil in/k\rceil, \ldots, j + \lceil (k-1)n/k\rceil\}$$

is a circular clique $K_{n'/k'}$.

Lemma 2.2 follows by coloring P^j with color jk' in $\mathbb{Z}/n'\mathbb{Z}$ and observing that the edges are precisely between the claimed polygons (see, for instance, Lemma 2.5).

2.3 A circular coloring matching the lower bound

We show Theorem 1.1 using the following auxiliary technical lemma and its consequences.

Lemma 2.3 *Let $y_1, \ldots, y_k \in \mathbb{R}_{\geq 0}$ and let $\sum_{i=1}^{k} y_i = z$. Then there exists a $j_0 \in [k]$ such that for all $m \in [k]$, $\sum_{i=j_0}^{j_0+m-1} y_i \geq mz/k$, where the indices are taken modulo k. Moreover, either there exists an $m' \in [k]$ for which $\sum_{i=j_0}^{j_0+m'-1} y_i > m'z/k$, or $y_i = z/k$ for each $i \in [k]$.*

A pigeonhole argument combined with an induction on k shows Lemma 2.3.

For a k-polygon $P = \{1 \leq y_1 < \ldots < y_k \leq n\}$ on the circle with points $1, \ldots, n$ in clockwise order, define the k-tuple of distances between the consecutive points $s(P) := (y_2 - y_1, y_3 - y_2, \ldots, y_1 - y_k + n) \in \mathbb{Z}^k_{\geq 1}$. We call $s(P)$ the *shape* of P. We say that the k-polygon with points $y_1 + 1, \ldots, , y_k + 1$ is obtained from P by a clockwise rotation of 1 (the addition is modulo n). For $i \geq 0$, the k-polygon obtained by rotating clockwise i times is denoted by $\rho_i(P)$.

The following corollary of Lemma 2.3 shows that every polygon can be rotated to contain n and such that the i-th point on the polygon is at a distance larger or equal than $\lceil in/k \rceil$ to the point n (in the counterclockwise direction).

Corollary 2.4 *Let $P = \{1 \le y_1 < ... < y_k \le n\}$ be a k-polygon and write $s(P) = (d_1, ..., d_k)$ for the shape of P. Then there is a $j_0 \in [k]$ such that for $i' = n - y_{j_0}$ we have*

$$n \in \rho_{i'}(P) \ \textit{ and } \ |\rho_{i'}(P) \cap \{1, \ldots, \lfloor mn/k \rfloor\}| \le m \ \textit{ for all } \ m \in [k].$$

Additionally, for $m \in [k]$ we have

$$|\rho_{i'}(P) \cap \{1, \ldots, \lfloor mn/k \rfloor\}| = m \iff \lfloor mn/k \rfloor = mn/k \textit{ and } \sum_{i=j_0}^{j_0+m-1} d_i = mn/k.$$

For a vector $d = (d_1, .., d_k) \in \mathbb{Z}^k_{\ge 2}$ with $\sum_{i=1}^k d_i = n$, let P°_d be the k-polygon with $s(P^\circ_d) = d$ and containing the point n. Note that P°_d is admissible. The set of k-polygons of the form P°_d (for some d as above) is an independent set in $\mathrm{IG}_{n,k}$. Define

$$\mathcal{L}_{n,k} := \{P^\circ_d \mid d = (d_1, ..., d_k) \in \mathbb{Z}^k_{\ge 2} \text{ and } \sum_{i=1}^t d_i \ge tn/k \text{ for all } t \in [k]\} .$$

The next lemma summarizes the main properties of the polygons in $\mathcal{L}_{n,k}$.

Lemma 2.5 *For any $j, i \in [0, n]$, $\{\rho_j(\mathcal{L}_{n,k}), \rho_{j+\lfloor in/k \rfloor}(\mathcal{L}_{n,k})\}$ and $\{\rho_j(\mathcal{L}_{n,k}), \rho_{j+\lceil in/k \rceil}(\mathcal{L}_{n,k})\}$ are independent sets, where $\rho_i(\mathcal{L}_{n,k}) = \{\rho_i(Q) \mid Q \in \mathcal{L}_{n,k}\}$.*

Indeed, if $P \in \mathcal{L}_{n,k}$ and $Q \in \rho_{\lfloor in/k \rfloor}(\mathcal{L}_{n,k})$ (resp. $Q \in \rho_{\lceil in/k \rceil}(\mathcal{L}_{n,k})$), then between n and $\lceil in/k \rceil$ (resp. $\lfloor in/k \rfloor$) Q contains $i + 1$ point. On the other side, either P contains at most i point between n and $\lceil in/k \rceil$ (resp. $\lfloor in/k \rfloor$), or both share the poin in/k if $\lfloor in/k \rfloor = in/k$ (resp. they share $\lfloor in/k \rfloor$) if P also contains $i + 1$ in such interval. The same argument applies if both sets $\mathcal{L}_{n,k}$ and $\rho_{\lfloor in/k \rfloor}(\mathcal{L}_{n,k})$ (or $\rho_{\lceil in/k \rceil}(\mathcal{L}_{n,k})$) are rotated by ρ_j.

The remaining part of the argument to show Theorem 1.1 can now be sketched. Any polygon is the rotation of a polygon in $\mathcal{L}_{n,k}$, by Corollary 2.4. The map χ colors the polygon P with $i \cdot k \in \mathbb{Z}/n\mathbb{Z}$ if i is the minimal index in $[0, n-1]$ such that $P \in \rho_i(\mathcal{L}_{n,k})$. Lemma 2.5 shows that the coloring χ is indeed a circular coloring with n/k colors.

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