Autoparatopism stabilized colouring games on rook's graphs

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Abstract

We introduce the autoparatopism variant of the autotopism stabilized colouring game on the $n \times n$ rook's graph as a natural generalization of the latter so that each board configuration is uniquely related to a partial Latin square of order n that respects a given autoparatopism $(\theta; \pi)$. To this end, we distinguish between $\pi \in \{\mathrm{Id}, (12)\}$ and $\pi \in \{(13), (23), (123), (132)\}$. The complexity of this variant is examined by means of the autoparatopism stabilized game chromatic number. Some illustrative examples and results are shown.

Keywords: Graph colouring game, partial Latin square, autoparatopism.

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1 Introduction

The *colouring game* [4,9] is played on a finite graph G. During the game, two players (Alice and Bob, with Alice playing first) alternately colour exactly one uncoloured vertex of G with a colour taken from a given palette so that none two adjacent vertices have the same colour. Alice wins if all the vertices are coloured at the end of the game; otherwise, Bob wins. The *game chromatic number* $\chi_g(G)$ is the least integer k for which Alice has a winning strategy in case of dealing with a palette of k colours. As the game may change significantly when Bob begins, it has been distinguished [1,2] between the game chromatic numbers $\chi_{g_A}(G)$ and $\chi_{g_B}(G)$ depending, respectively, on whether Alice or Bob begins the game.

In 2011, Schlund [13] dealt with colouring games on $n \times n$ rook's graphs, during which every board configuration corresponds to a *partial Latin square* of order n, that is, an $n \times n$ array $L = (l_{ij})$ in which each cell (i, j) is either empty or contains one symbol chosen from the set $[n] := \{1, \ldots, n\}$ so that each symbol occurs at most once in each row and in each column. This is uniquely related to a partial colouring of a labeled $n \times n$ rook's graph, and uniquely determined by its *entry set* $\mathrm{Ent}(L) := \{(i, j, L[i, j]) \in [n] \times [n] \times [n]\}$. Any triple $(i, j, L[i, j]) \in \mathrm{Ent}(L)$ is called an *entry* of L.

Let S_n and $\mathrm{PLS}(n)$ respectively denote the symmetric group on $[n]$ and the set of partial Latin squares of order n. For each $L \in \mathrm{PLS}(n)$, every tuple $\Theta = (\alpha, \beta, \gamma; \pi) \in (S_n \times S_n \times S_n) \rtimes S_3$ gives rise to $L^\Theta \in \mathrm{PLS}(n)$ such that

$$\mathrm{Ent}(L^\Theta) = \{(\alpha(e_{\pi(1)}), \beta(e_{\pi(2)}), \gamma(e_{\pi(3)})) \mid (e_1, e_2, e_3) \in \mathrm{Ent}(L)\}.$$

The tuple Θ is called a *paratopism* of $\mathrm{PLS}(n)$. This is an *isotopism* if $\pi = \mathrm{Id}$ is the trivial permutation in S_3. Further, if $L^\Theta = L$, then Θ is called an *autoparatopism* of L (an *autotopism*, if $\pi = \mathrm{Id}$). Motivated by the current growing research on autotopisms of partial Latin squares and related structures [5,6,8,10,11,14,15], the authors [3,7] have recently generalized the colouring game proposed by Schlund by introducing the (α, β, γ)-*stabilized colouring game* on the $n \times n$ rook's graph so that each board configuration corresponds to a partial Latin square having a given triple $(\alpha, \beta, \gamma) \in S_n \times S_n \times S_n$ as an autotopism. In this paper, we introduce the autoparatopism variant of this last colouring game so that each board configuration is uniquely related to a partial Latin square having a given tuple $(\alpha, \beta, \gamma; \pi) \in (S_n \times S_n \times S_n) \rtimes S_3$ as an autoparatopism. The autotopism stabilized colouring game arises when $\pi = \mathrm{Id}$. See [8,12,14] for recent studies and applications on autoparatopisms.

2 Cell orbits

Let $\mathfrak{A}_n := (S_n \times S_n \times S_n) \rtimes S_3$. The action of a paratopism $\Theta = (\alpha, \beta, \gamma; \pi) \in \mathfrak{A}_n$ over a triple $(e_1, e_2, e_3) \in [n] \times [n] \times [n]$ gives rise to the triple

$$\Theta((e_1, e_2, e_3)) := (\alpha(e_{\pi(1)}), \beta(e_{\pi(2)}), \gamma(e_{\pi(3)})). \tag{1}$$

Let p_t denote the projection over the t^{th} coordinate. We define the *k-cell orbit* of a pair $(i, j) \in [n] \times [n]$ under the action of Θ as

$$\mathfrak{o}_{k;\Theta}((i, j)) := \{(p_1(\Theta^m((i, j, k))), p_2(\Theta^m((i, j, k)))) \mid m \in \mathbb{N}\}. \tag{2}$$

Then, we define the *cell orbit* of the pair (i, j) under the action of Θ as

$$\mathfrak{o}_{\Theta}((i, j)) := \bigcup_{k \in [n]} \mathfrak{o}_{k;\Theta}((i, j)). \tag{3}$$

This generalizes the concept of cell orbit [3, 7, 12, 15] concerning those cases where $\pi \in \{\mathrm{Id}, (12)\}$. In such cases, the third component of the triple under consideration does not play any role, and hence, all the sets in (3) coincide. This does not happen in general when $\pi \in \{(13), (23), (123), (132)\}$, as the following example illustrates.

Example 2.1 Let $\Theta = ((123), (123), (123); (123)) \in \mathfrak{A}_3$. Then, $\mathfrak{o}_{\Theta}((1, 2)) = \{(1, 1), (1, 2), (1, 3), (2, 2), (3, 2)\}$. To see it, observe the following three partial Latin squares of order three, all of them having Θ as an autoparatopism. They determine the 1-, 2- and 3-cell orbits of the pair $(1, 2)$ under the action of Θ.

	1	3
	3	

3	2	
	3	

	3	

$\lhd$

3 Description of the colouring game

Let $\Theta = (\alpha, \beta, \gamma; \pi) \in \mathfrak{A}_n$. The Θ-*stabilized colouring game* is played by two players, Alice and Bob, on the empty partial Latin square $L \in \mathrm{PLS}(n)$ as board, by using $n' \geq n$ colours. Alternately, they choose an empty cell (i, j) in the board and a symbol $k \in [n']$, and colour the former by the latter by setting $(i, j, k) \in E(L)$. The resulting configuration must obey the next rules.

(i) It must be a partial Latin square.

(ii) It must be Θ-*compatible*, that is, for each $m \in \mathbb{N}$, either $\Theta^m((i,j,k)) \in \mathrm{Ent}(L)$ or the cell $(p_1(\Theta^m((i,j,k))), p_2(\Theta^m((i,j,k))))$ is empty.

(iii) For each $m \in \mathbb{N}$, there does not exist an entry $(p_1(\Theta^m((i,j,k))), j', p_3(\Theta^m((i,j,k))))$ or $(i', p_2(\Theta^m((i,j,k))), p_3(\Theta^m((i,j,k))))$ in the entry set $\mathrm{Ent}(L)$, where $j' \neq p_2(\Theta^m((i,j,k)))$ and $i' \neq p_1(\Theta^m((i,j,k)))$.

As in the conventional colouring game, Alice wins if all the cells of the board are filled at the end of the game; otherwise, Bob wins. Similarly to the autotopism stabilized colouring game described in [3,7], not every paratopism can be used to describe a well-defined autoparatopism stabilized colouring game.

Firstly, the paratopism Θ must be *feasible*. That is, for each triple $(i,j,k) \in [n] \times [n] \times [n]$, if there exist two positive integers $m_1, m_2 \in \mathbb{N}$ such that $p_t(\Theta^{m_1}((i,j,k))) = p_t(\Theta^{m_2}((i,j,k)))$, for both $t \in \{1,2\}$, then this condition also holds for $t = 3$. This enables the first player to fill, as first move, any cell (i,j) of the board with any given symbol $k \in [n]$ so that the colouring of every cell within the k-cell orbit $\mathfrak{o}_{k;\Theta}((i,j))$ is uniquely determined by the action of Θ over the entry (i,j,k).

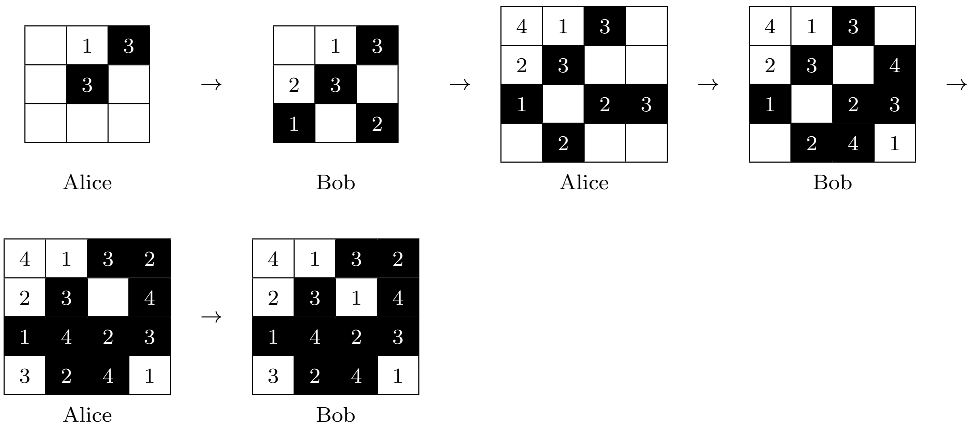
Secondly, Θ must be *extendable*. That is, the paratopism $\Theta' = (\alpha', \beta', \gamma'; \pi) \in \mathfrak{A}_{n'}$ must be feasible, for all $n' > n$ and $\alpha', \beta', \gamma' \in S_{n'}$, where

- If $\pi \in \{\mathrm{Id}, (12)\}$, then $\alpha' = \alpha$; $\beta' = \beta$; and $\gamma'(k) = \gamma(k)$, if $k \leq n$, and $\gamma'(k) = k$, otherwise.
- If $\pi \in \{(13), (23), (123), (132)\}$, then, for each $\delta \in \{\alpha, \beta, \gamma\}$, we have that $\delta'(k) = \delta(k)$, if $k \leq n$, and $\delta'(k) = k$, otherwise.

This enables both players to use any previously stipulated number $n' \geq n$ of colours during the game without breaking the rule of extendability.

Remark that, if $\pi \in \{\mathrm{Id}, (12)\}$, then the order of the board remains the same during all the game. This is, however, not true in case of being $\pi \in \{(13), (23), (123), (132)\}$. In such cases, the order of the board is increased one unit for each extra colour $k > n$ that is used by the players. In practice, we can suppose that this increase only takes place at the precise moment in which Alice uses a colour $k > n$ that has not been used before, and that she is always the first player who can use such extra colours. Otherwise, we can suppose Θ' to be the initial paratopism. Even if this modification of the original board is allowed by none currently known graph colouring game, during which edges and vertices of the graph under consideration do not change, the study of this variant could give rise to new open questions to deal with.

Example 3.1 Let Θ be the paratopism defined in Example 2.1. The following sequence of configurations describes a possible development of a Θ-autotopism stabilized colouring game. Keeping in mind the *passing board* technique described in [7], each move is represented by (a) filling a cell (i, j) with a symbol k; (b) colouring in black the background of the rest of cells within the k-cell orbit $\mathfrak{o}_{k;\Theta}((i, j))$; and (c) writing the symbol that these cells should contain according to the second rule of the game.



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As in the conventional colouring game, the complexity of each Θ-stabilized game chromatic number is examined by means of its Θ-*stabilized game chromatic number*, which consists of the least positive integer $\chi_{g_A}^{\Theta}$ for which Alice has a winning strategy. Similarly to the classical colouring game, we distinguish the variant $\chi_{g_B}^{\Theta}$ in case of being Bob who starts the game. This gives rise to the following question.

Problem 3.2 *Do both numbers, $\chi_{g_A}^{\Theta}$ and $\chi_{g_B}^{\Theta}$, exist for any feasible and extendable paratopism?*

As a first stage to deal with this question, we illustrate both numbers for some feasible and extendable paratopisms of partial Latin squares of small order.

Theorem 3.3 *The following results hold.*

a) If $n = 2$, then $\chi_g^{(\mathrm{Id},(12),(12);\pi)} = 2$, for all $g \in \{g_A, g_B\}$ and $\pi \in S_3$.

b) If $n = 2$, then $\chi_g^{(\mathrm{Id},\mathrm{Id},\mathrm{Id};\pi)} = 2$, for all $g \in \{g_A, g_B\}$ and $\pi \in S_3 \setminus \{\mathrm{Id}\}$.

c) If $n = 3$, then $\chi_g^{((123),(123),(123);\pi)} = 3$, for all $g \in \{g_A, g_B\}$ and $\pi \in S_3$.

d) If $n = 3$, then $\chi_{g_A}^{(\mathrm{Id},\mathrm{Id},\mathrm{Id};(12))} = 4 > 3 = \chi_{g_B}^{(\mathrm{Id},\mathrm{Id},\mathrm{Id};(12))}$.

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