Dyck-Eulerian digraphs

Pietro Codara [1,2] Ottavio M. D'Antona [3]

Dipartimento di Informatica
Università degli Studi di Milano
Via Comelico 39, I-20135 Milan, Italy

Abstract

We introduce a family of Eulerian digraphs, $\mathscr{E}$, associated with Dyck words. We provide the algorithms implementing the bijection between $\mathscr{E}$ and $\mathscr{W}$, the set of Dyck words. To do so, we exploit a binary matrix, that we call *Dyck matrix*, representing the cycles of an Eulerian digraph.

Keywords: Dyck Word, Eulerian Digraph, Cycle-Vertex Incidence Matrix

1 Introduction and basic notions

A digraph G is *Eulerian* if at every vertex the in-degree equals the out-degree. (Note that we do not require G to be connected.) The edge set of an Eulerian digraph G can be partitioned into directed cycles.

A *Dyck word* on the alphabet $\{U, D\}$ is a string with the same number of U's and D's, and such that the number of U's in any initial segment is greater or equal to the number of D's. We denote by $\mathscr{W}$ the set of all Dyck words on

[1] Pietro Codara is supported by the INdAM-Marie Curie Cofund project *LaVague* (FP7-PEOPLE-2012-COFUND 600198).

[2] Email: `codara@di.unimi.it`

[3] Email: `dantona@di.unimi.it`

the alphabet $\{U, D\}$. A *Dyck path* is a lattice path in $\mathbb{Z}^2$ starting at $(0, 0)$, ending on the x-axis, with unit steps $(1, 1)$ and $(1, -1)$, and such that it never passes below the x-axis. One can easily build a correspondence between Dyck paths and Dyck words, by mapping a $(1, 1)$ step to the character U and a $(1, -1)$ step to the character D. Fig. 1 shows the Dyck path associated with the word $UUUDUDDDUD$.

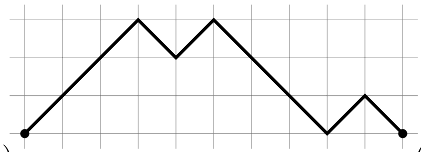

Fig. 1: The Dyck path $UUUDUDDDUD$.

The goal of this work is to establish a bijection between $\mathscr{W}$ and a family $\mathscr{E}$ of Eulerian digraphs. Specifically, the bijection is between elements of $\mathscr{W}$ with $2k$ steps and elements in $\mathscr{E}$ with k vertices. (Hence the elements of $\mathscr{E}$ with k vertices are as many as the k^{th} Catalan number.) In Section 2 we introduce $\mathscr{E}$, together with a family $\mathscr{M}$ of *Dyck matrices*, and we describe a one-to-one correspondence between $\mathscr{M}$ and $\mathscr{E}$. In Section 3 we establish a bijection between $\mathscr{M}$ and $\mathscr{W}$, and we provide algorithms implementing such bijection. In conclusion, we highlight the connection between Dyck matrices and other combinatorial objects.

2 Dyck-Eulerian digraphs and Dyck matrices

For a vector $\mathbf{s} = (s_1, s_2, \ldots, s_n)$ of positive integers, we call an Eulerian digraph $\mathbf{s}$-*labelled* if its edge set is partitioned into n directed cycles of length s_1, s_2, $\ldots$, s_n, each with a distinguished first edge (and hence a unique second, third, etc. edge) [4]. We refer to $\mathbf{s}$ as the *signature* of the graph. Fig. 2 shows a $(3, 3, 1)$-labelled Eulerian digraph, with its 3 directed cycles of size 3, 3 and 1; the j^{th} edge of the i^{th} cycle is labelled $e_{i,j}$. (Note that we allow parallel edges.)

We fix an order on the vertices of $\mathbf{s}$-labelled Eulerian digraphs, in accordance with the order between cycles. Since cycles are rooted, an order on the vertices of the same cycle is naturally established. If $C_1, \ldots, C_n$ are the (ordered) cycles of an $\mathbf{s}$-labelled Eulerian digraphs E, we label the vertices of E inductively, as follows. The r vertices of C_1 are labelled $v_1, \ldots, v_r$, according with the order on the edges. Suppose that we have labelled all the vertices

[4] Similar digraphs are investigating in [2]. Notice that in that case $\mathbf{s}$ is a multiset, *i.e.* the order among cycles does not matter.

$v_1, \ldots, v_s$ of the cycles $C_1, \ldots, C_p$. The t vertices of C_{p+1} that are not vertices of C_{p-1} are labelled $v_{s+1}, \ldots, v_{s+t}$, according with the order on the edges.

Definition 2.1 Let E be an $\mathbf{s}$-labelled Eulerian digraph, with signature $\mathbf{s} = (s_1, s_2, \ldots, s_n)$ and cycles $C_1, C_2, \ldots, C_n$. We say that E is a *Dyck-Eulerian digraph*, or D-E graph, for short, if and only if the following conditions hold.

(E1) No cycle shares all its vertices with another cycle.

(E2) If two cycles C_i and C_{i+1} of E share k vertices, these must be the first k vertices of both cycles.

(E3) If a vertex of C_{i+1} is not a vertex of C_i, then it is not a vertex of $C_1, \ldots, C_{i-1}$.

We denote by $\mathscr{E}$ the set of all D-E graphs.

One can easily check that the graph in Fig. 2 is a D-E graph.

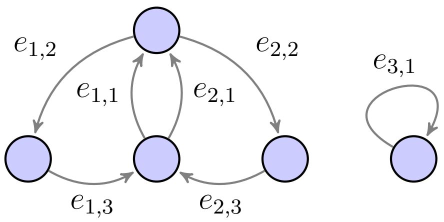

Fig. 2: A $(3,3,1)$-labelled Eulerian digraph.

We will construct the promised bijection between $\mathscr{E}$ and $\mathscr{W}$ exploiting a binary matrix, defined as follows.

Definition 2.2 A binary matrix $M = (m_{i,j})$ of size $n \times k$, is a *Dyck matrix* if it is the empty matrix — written $(\ \ )$ —, or it satisfies the following conditions.

(M1) There exists $0 < h \leq k$ such that $m_{1,j} = 1$ if and only if $j \leq h$.

(M2) For each $1 < i \leq n$, there exist $1 \leq a_i \leq b_i < c_i \leq k$ such that a_i is the smallest index satisfying $m_{i-1,a_i} = 1$ and $m_{i,a_i} = 0$, b_i is the greatest index such that $m_{i-1,b_i} = 1$, and c_i is the greatest index such that $m_{i,c_i} = 1$. Moreover, the following hold:
(M2.1) $m_{i,j} = m_{i-1,j}\,,$ for $j = 1, \ldots, a_i - 1$;
(M2.2) $m_{i,j} = 0\,,$ for $j = a_i, \ldots, b_i$;
(M2.3) $m_{i,j} = 1\,,$ for $j = b_i + 1, \ldots, c_i$;
(M2.4) $m_{i,j} = 0\,,$ for $j = c_i + 1, \ldots, k$.

(M3) $m_{n,k} = 1$.

We denote by $\mathscr{M}$ the set of all Dyck matrices.

Example 2.3 Consider the following matrices:

$$M = \begin{pmatrix} 1\,1\,1\,0\,0 \\ 1\,1\,0\,1\,0 \\ 0\,0\,0\,0\,1 \end{pmatrix}, \qquad X = \begin{pmatrix} 1\,1\,1\,0\,0 \\ 1\,0\,0\,1\,1 \\ 0\,0\,0\,0\,1 \end{pmatrix}. \tag{1}$$

The matrix M is a Dyck matrix. Conditions **(M1)** and **(M2)** are verified by setting $h = 3$, $a_2 = 2$, $b_2 = 3$, $c_2 = 4$, $a_3 = 1$, $b_3 = 4$, and $c_3 = 5$. On the other hand, X is not a Dyck matrix. Indeed, there are no a_3, b_3, and c_3, with $b_3 < c_3$, satisfying condition **(M2)**.

Each $\mathbf{s}$-labelled Eulerian digraph with n cycles and k vertices can be represented by a binary matrix $M = (m_{i,j})$ of size $n \times k$, where rows represent the cycles, in order, columns represent the vertices, in order, and $m_{i,j} = 1$ if and only if v_j is a vertex of C_i. The matrix M is the cycle-vertex incidence matrix of E. One can easily check that the matrix M in (1) is the cycle-vertex incidence matrix of the D-E graph in Fig. 2.

A one-to-one correspondence between $\mathscr{E}$ and $\mathscr{M}$ holds:

Proposition 2.4 *An* $\mathbf{s}$*-labelled Eulerian digraph* E *is a D-E graph if and only if its cycle-vertex incidence matrix is a Dyck matrix.*

Proof. We sketch the proof. We observe that Condition (M2.3) guarantees **(E1)**, Conditions (M2.1) and (M2.2) guarantee **(E2)** and **(E3)**, and (M2.1), together with (M2.3) and **(M1)**, bring the correct order on the vertices of the graph.

On the other side, take a D-E graph on k vertices $v_1, \ldots, v_k$, with cycles $C_1, \ldots, C_n$, and consider its cycle-vertex incidence matrix M. Since the first h vertices are the vertices of C_1, condition **(M1)** holds on M. By **(E1)**, for each $i = 2, \ldots, n$ there exist $b_i < c_i \leq k$ satisfying (M2.3). Conditions **(E2)** and **(E3)** guarantee the existence of $a_i \leq b_i$ satisfying (M2.1) and (M2.2). Conditions (M2.4) and **(M3)** are satisfied by construction. □

3 Dyck matrices and Dyck words

3.1 From Dyck words to Dyck matrices

We supply an online algorithm that converts a Dyck word into a Dyck matrix. The basic idea is that we can split a Dyck word into *slopes*, *i.e.* maximal continuous sequences of U's, and *descents*, *i.e.* maximal continuous sequences of D's. A *peak* is a slope followed by a descent. Every peak represents a cycle:

the number of U's from the beginning of the slope to its end is the number of new vertices (not shared with the previous cycles), while the number of vertices shared with the subsequent cycle is given by the difference between the total number of U's and D's (since the beginning of the word). Our algorithm is input by a stream of U's and D's forming a Dyck word, and outputs a Dyck matrix, performing the following steps.

getMatrix(s) — Input: a Dyck word $\mathtt{s}$ of U's and D's

1: $\mathtt{U} := 0$, $\mathtt{D} := 0$ (counters for U's and D's), $\mathtt{k} := 0$ (number of vertices shared by two consecutive cycles), $\mathtt{p} := U$ (previous character), $\mathtt{c} := \mathtt{EOS}$ (current character, set to End Of String), $\mathtt{R} := ()$ (a binary vector), $\mathtt{M} := ()$ (a binary matrix).

2: Read the next character and store it in $\mathtt{c}$.

3: **If** $\mathtt{c} = D$, then increment $\mathtt{D}$ and set $\mathtt{p} := D$.

4: **Else If** $\mathtt{c} = U$ do the following.

4.1: **If** $\mathtt{p} = U$, then increment $\mathtt{U}$.
4.2: **Else**, do the following.
4.2.1: Modify $\mathtt{R}$: maintain the first $\mathtt{k}$ 1's and reset the others elements to 0.
4.2.2: Append to $\mathtt{R}$ a sequence of $\mathtt{U} - \mathtt{k}$ 1's.
4.2.3: Append $\mathtt{R}$ to $\mathtt{M}$, as a new row.
4.2.4: Fill the previews rows of $\mathtt{M}$ with 0's, till their length equals that of $\mathtt{R}$.
4.2.5: Set $\mathtt{p} = U$, $\mathtt{k} = \mathtt{U}{-}\mathtt{D}$, $\mathtt{U} = \mathtt{k}{+}1$, $\mathtt{D} = 0$.

5: **Repeat** from Step 2, until there are no characters left to read ($\mathtt{c} = \mathtt{EOS}$).

6: Add the last row to $\mathtt{M}$: perform Steps 4.2.1–4.2.4.

7: **Return** $\mathtt{M}$.

Note that if we run the algorithm on the word $UUUDUDDDUD$ we obtain the matrix M in (1), that is the Dyck matrix associated with the D-E graph in Fig. 2. The intermediate steps of the execution are illustrated in Fig. 3.

$$\begin{array}{ccc} UUUD & UUUDUDDD & UUUDUDDDUD \\ \begin{pmatrix} 1 & 1 & 1 \end{pmatrix} & \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix} & \begin{pmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

Fig. 3. From $UUUDUDDDUD$ to the associated Dyck matrix, step by step.

3.2 From Dyck matrices to Dyck words

The main idea to convert a Dyck matrix into a Dyck word is to scan the whole matrix using a vertical two-value sliding window (a mask showing two elements of the matrix one on top of the other).

`getWord(M=(m_{i,j}))` — Input: a binary Matrix `M` of size $n \times k$

1: `w` := “ ” (string of U's and D's).

2: Insert two rows of 0's in `M`, one at the beginning and one at the end.

3: **For** `i` $:= 1$ **to** $n+1$

  3.1: **For** `j` $:= 1$ **to** `k`

    3.2.1: **If** $(\mathrm{m}_{\mathrm{i,j}},\mathrm{m}_{\mathrm{i+1,j}}) = (0,1)$, then append U to the word `w`.

    3.2.2: **If** $(\mathrm{m}_{\mathrm{i,j}},\mathrm{m}_{\mathrm{i+1,j}}) = (1,0)$, then append D to the word `w`.

4: **Return** `w`.

3.3 Main result

Showing that the algorithms described in Sections 3.1 and 3.2 are correct, in that they associate a Dyck word with a Dyck matrix, and viceversa, and that the algorithms are the inverse of each other, we can prove the following.

Theorem 3.1 *$\mathscr{W}$ and $\mathscr{M}$ are in bijection.*

4 Conclusion and future work

In [3], the authors provide a combinatorial interpretation of the Stirling numbers (of the second kind) of a Dyck word w in terms of stable partitions of a graph associated with w. Using some ideas from [2], we plan to describe the same numbers in terms of suitable transformations on an Eulerian digraphs, in fact a D-E graph, associated with w.

Acknowledgments. We thank M. Genuzio for his work in programming the algorithms in [1], of which the present note is an improved version.

References

[1] Codara, P., O. M. D'Antona and M. Genuzio, *Eulerian digraphs and Dyck words, a bijection*, GASCom 2014, Bertinoro, Italy, arXiv:1407.2461 [cs.DM] (2014).

[2] Codara, P., O. M. D'Antona and P. Hell, *A simple combinatorial interpretation of certain generalized Bell and Stirling numbers*, Discrete Mathematics **318** (2014), pp. 53–57.

[3] Engbers, J., D. Galvin and J. Hilyard, *Combinatorially interpreting generalized stirling numbers*, European Journal of Combinatorics **43** (2015), pp. 32 – 54.