Edges incident with a vertex of degree greater than four and a lower bound on the number of contractible edges in a 4-connected graph

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Abstract

In this paper, we prove that the number of 4-contractible edges (edges that after contraction do not change the connectivity of the initial graph) of a 4-connected graph G is at least $(1/28)\sum_{x\in V_{\geq 5}(G)} \deg_G(x)$, where $V_{\geq 5}(G)$ denotes the set of those vertices of G which have degree greater than or equal to 5.

This is the refinement of the result proved by Ando et al. [On the number of 4-contractible edges in 4-connected graphs, *J. Combin. Theory Ser. B* **99** (2009) 97–109].

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1 Introduction

In this paper, we consider finite simple graphs, which have neither loops nor multiple edges. For terminology and notation not defined in this paper, we refer the reader to [4].

Let $G = (V(G), E(G))$ be a graph. For $x \in V(G)$, $N_G(x)$ denotes the neighborhood of x and $\deg_G(x)$ denotes the degree of x; thus $\deg_G(x) = |N_G(x)|$. For $e \in E(G)$, we let $V(e)$ denote the set of endvertices of e. The complete graph of order n is denoted by K_n. The complete bipartite graph $K_{1,n}$ with partite sets of cardinalities 1 and n is called a *star*. For a graph H, let nH denote the graph with n components, each isomorphic to H. For an integer $i \geq 0$, we let $V_i(G)$ denote the set of vertices x of G with $\deg_G(x) = i$ and we let $V_{\geq i}(G) = \cup_{j \geq i} V_j(G)$. A subset S of $V(G)$ is called a *cutset* if $G - S$ is disconnected. For an integer $k \geq 1$, we say that G is k-connected if $|V(G)| \geq k + 1$ and G has no $(k-1)$-cutset.

Let G be a 4-connected graph. For $e \in E(G)$, we let G/e denote the graph obtained from G by contracting e into one vertex (and replacing each resulting pair of double edges by a simple edge). We say that e is 4-*contractible* or 4-*noncontractible* according as G/e is 4-connected or not. A 4-noncontractible edge $e = ab$ is said to be *trivially* 4-*noncontractible* if there exists a vertex z of degree 4 such that $za, zb \in E(G)$. We let $E_c(G)$, $E_n(G)$ and $E_{tn}(G)$ denote the set of 4-contractible edges, the set of 4-noncontractible edges and the set of trivially 4-noncontractible edges, respectively.

The following characterization of 4-connected graphs with $E_c(G) = \emptyset$ was obtained by Fontet and independently by Martinov.

Theorem A (Fontet [7]; Martinov [10]) *Let G be a 4-connected graph of order n with $E_c(G) = \emptyset$. Then one of the following holds:*

(1) *G is the square of the cycle of order n; i.e., we can write $V(G) = \{v_1, v_2, \ldots, v_n\}$ so that $E(G) = \{v_i v_j \mid i - j \in \{\pm 1, \pm 2\} \pmod{n}\}$; or*

(2) *there exists a 3-regular graph H such that G is the line graph of H.*

In view of Theorem A, it is natural to expect that one can estimate $|E_c(G)|$ in terms of degrees of vertices of G, and also in terms of the number of edges of G not contained in a triangle. Along this line, the following results have been obtained.

Theorem B (Ando, Egawa, Kawarabayashi and Kriesell [3]) *If G is a 4-connected graph, then $|E_c(G)| \geq (1/68) \sum_{u \in V(G)} (\deg_G(u) - 4)$.*

Theorem C (Ando and Egawa [1]) *If G is a 4-connected graph, then* $|E_c(G)| \geq |V_{\geq 5}(G)|$.

Further we let $\tilde{E}(G)$ denote the set of those edges of G which are not contained in a triangle. Let $\tilde{V}$ denote the set of those vertices of G which are incident with an edge in $\tilde{E}(G) \cap E_n(G)$, and let $\hat{G}$ denote the subgraph of G induced by the edge set $\tilde{E}(G) \cap E_n(G)$; that is to say, $\tilde{V} = \cup_{e \in \tilde{E}(G) \cap E_n(G)} V(e)$ and $\hat{G} = (\tilde{V}, \tilde{E}(G) \cap E_n(G))$. Finally we let Y^* denote the graph of order 6 defined by $V(Y^*) = \{w, z\} \cup \{v_i \,|\, 1 \leq i \leq 4\}$, $\quad E(Y^*) = \{wz, v_1w, v_2w, v_3z, v_4z\}$.

Theorem D (Ando and Egawa [2]) *Let G be a 4-connected graph, and suppose that $|\tilde{E}(G)| \geq 15$. Then $|E_c(G)| \geq (|\tilde{E}(G)| + 8)/4$.*

In Theorems C and D, the lower bound on $|E_c(G)|$ is best possible. However, the bound 15 on $|\tilde{E}(G)|$ in the assumption of Theorem D is not best possible. In fact, the following theorem concerning the refinements of Theorem D has already been proved.

Theorem E (Egawa et al. [5,6]; Kotani et al. [9]; Nakamura [11]) *Let G be a 4-connected graph, and suppose that $1 \leq |\tilde{E}(G)| \leq 14$. Then $|E_c(G)| \geq (|\tilde{E}(G)| + 4)/4$. Further we have $|E_c(G)| \geq (|\tilde{E}(G)| + 8)/4$ unless one of the following holds*:

(1) $|\tilde{E}(G)| = 1$ *and* $\hat{G} = K_2$;

(2) $|\tilde{E}(G)| = 2$ *and* $\hat{G} = \emptyset$;

(3) $|\tilde{E}(G)| = 3$ *and* $\hat{G} = K_2$;

(4) $|\tilde{E}(G)| = 4$ *and* $\hat{G} = 2K_2$;

(5) $|\tilde{E}(G)| = 5$ *and* $\hat{G} = 2K_2$ *or* $K_{1,2}$;

(6) $|\tilde{E}(G)| = 6$ *and* $\hat{G} = 3K_2$; *or*

(7) $|\tilde{E}(G)| = 9$ *and* $\hat{G} = Y^*$.

In Theorem B, the coefficient 1/68 seems far from best possible. The purpose of this paper is to prove the following theorem which is the refinement of Theorem B.

Theorem 1 *If G is a 4-connected graph, then*

$$|E_c(G)| \geq \frac{1}{28} \sum_{u \in V_{\geq 5}(G)} \deg_G(u).$$

The coefficient 1/28 in Theorem 1 still seems not to be best possible. However we construct examples showing that the coefficient of Theorem 1 is at most 1/13.

The organization of this paper is as follow. In Section 2, we introduce a known result proved in [8] and introduce some lemmas for the proof of Theorem 1. Finally we prove Theorem 1 in Section 3.

2 Preliminaries

Throughout the rest of this paper, we let G be a 4-connected graph. Let L be the set of edges e such that both endvertices of e have degree 4, and let $F = E_n(G) - E_{tn}(G) - L$. Also let $\tilde{V}(G)$ denote the set of those vertices of G which are incident with an edge in F, and let $\tilde{G}$ denote the spanning subgraph of G with edge set F; that is to say, $\tilde{V}(G) = \cup_{e \in F} V(e)$ and $\tilde{G} = (V(G), F)$. Set

$$\begin{aligned}\mathcal{L} = &\{(S, A) \mid S \text{ is a 4-cutset}, A \text{ is the union of the vertex set of} \\ &\text{some components of } G - S, \emptyset \neq A \neq V(G) - S\}.\end{aligned}$$

Now take $(S_1, A_1), \ldots, (S_k, A_k) \in \mathcal{L}$ so that for each $e \in F$, there exists S_i such that $V(e) \subseteq S_i$. We choose $(S_1, A_1), \ldots, (S_k, A_k)$ so that k is minimum and so that $(|A_1|, \cdots, |A_k|)$ is lexicographically minimum, subject to the condition that k is minimum. Set $\mathcal{S} = \{S_1, \ldots, S_k\}$.

For two distinct 4-cutset $S, T \in \mathcal{S}$, we say that S *crosses* T if S intersects with every component of $G - T$. Furthermore, we call $\mathcal{S}$ *is cross free* if any two members of $\mathcal{S}$ do not cross. The following lemma plays an important role in the proof of Theorem 1.

Lemma 2.1 (Kotani and Nakamura [8]) *Suppose that $|V(G)| \geq 9$ and some two members of $\mathcal{S}$ cross. Then there exists a 4-connected graph G' such that $|V(G')| = |V(G)| - 2$ and*

$$|E_c(G)| - |E_c(G')| \geq \max\left\{1, \frac{1}{10}\left(\sum_{x \in V_{\geq 5}(G)} \deg_G(x) - \sum_{x \in V_{\geq 5}(G')} \deg_{G'}(x)\right)\right\}.$$

We introduce two lemmas for the proof of Theorem 1. In order to introduce the first result, we set $R = \{(u, a) \mid ua \in E(G) - F, u \in V_{\geq 5}(G)\}$ and $Q = \{(x, y) \mid xy \in E_c(G)\}$. Then the following lemma holds.

Lemma 2.2 *Suppose that $\mathcal{S}$ is cross free. Then $|R| \leq 4|Q|$.*

We also set $J = \{(u, a) \mid ua \in F, u \in V_{\geq 5}(G)\}$. Now we introduce the second result which will be used in the proof of Theorem 1.

Lemma 2.3 *Suppose that $\mathcal{S}$ is cross free. Then $|J| \leq 10|Q|$.*

3 Proof of Theorem 1

In this section, we prove Theorem 1. If G is 4-regular, then $\sum_{x \in V_{\geq 5}(G)} \deg_G(x) = 0$, and hence the desired inequality holds immediately. Thus we may assume that G is not 4-regular, thus $|V_{\geq 5}(G)| \geq 1$. By way of contradiction, we suppose that $|E_c(G)| < (1/28) \sum_{x \in V_{\geq 5}(G)} \deg_G(x)$, thus $\sum_{x \in V_{\geq 5}(G)} \deg_G(x) > 28|E_c(G)|$. We may assume that we have chosen G such that $|V(G)|$ is as small as possible. Then the following claim holds.

Claim 3.1 $|V(G)| \geq 9$.

Proof. Suppose that $|V(G)| \leq 8$. Set $b(G) := |E(G)| - 2|V(G)|$. Then $b(G) = (1/2) \sum_{x \in V_{\geq 5}(G)} \deg_G(x) - 2|V_{\geq 5}(G)| > 14|E_c(G)| - 2|V_{\geq 5}(G)|$. Since G is a 4-connected graph such that G is not 4-regular, we have $6 \leq |V(G)| \leq 8$, and hence $b(G) = |E(G)| - 2|V(G)| \leq \max\{(15-12), (21-14), (28-16)\} = 12$. By Theorem C, $b(G) > 14|E_c(G)| - 2|V_{\geq 5}(G)| \geq 14|V_{\geq 5}(G)| - 2|V_{\geq 5}(G)| = 12|V_{\geq 5}(G)| \geq 12$, which contradicts $b(G) \leq 12$. □

Let $\mathcal{S}$ be as in Section 2. By making use of Claim 3.1, we prove the following claim.

Claim 3.2 *$\mathcal{S}$ is cross free.*

Proof. Suppose that some two members of $\mathcal{S}$ cross. By Lemma 2.1 and Claim 3.1, there exists a 4-connected graph G' such that $|V(G')| = |V(G)| - 2$ and $|E_c(G)| - |E_c(G')| \geq \max\{1, (1/10)(\sum_{x \in V_{\geq 5}(G)} \deg_G(x) - \sum_{x \in V_{\geq 5}(G')} \deg_{G'}(x))\}$. Since $|V(G')| < |V(G)|$, we have $|E_c(G')| \geq (1/28) \sum_{x \in V_{\geq 5}(G')} \deg_{G'}(x)$. Thus

$$1 \leq |E_c(G)| - |E_c(G')| < \frac{1}{28} \left(\sum_{x \in V_{\geq 5}(G)} \deg_G(x) - \sum_{x \in V_{\geq 5}(G')} \deg_{G'}(x) \right) < \frac{1}{10} \left(\sum_{x \in V_{\geq 5}(G)} \deg_G(x) - \sum_{x \in V_{\geq 5}(G')} \deg_{G'}(x) \right),$$

which is a contradiction. □

We are now in a position to complete the proof of Theorem 1. Note that $|R| + |J| = |\{(x, y) \mid xy \in E(G), x \in V_{\geq 5}(G)\}| = \sum_{x \in V_{\geq 5}(G)} \deg_G(x)$. It follows from Lemmas 2.2, 2.3 and Claim 3.2 that

$$\sum_{x \in V_{\geq 5}(G)} \deg_G(x) = |R| + |J| \leq 4|Q| + 10|Q| = 14|Q| = 28|E_c(G)|,$$

which contradicts the assumption that $|E_c(G)| < (1/28) \sum_{x \in V_{\geq 5}(G)} \deg_G(x)$. This completes the proof of Theorem 1. □

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