Elliptic semiplanes and regular graphs with girth 5

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Abstract

A (k, g)-graph is a k-regular graph with girth g and a (k, g)-cage is a (k, g)-graph with the fewest possible number of vertices. The *cage problem* consists of constructing (k, g)-graphs of minimum order $n(k, g)$. We focus on girth $g = 5$, where cages are known only for degrees $k \leq 7$. Considering the relationship between finite geometries and graphs we establish upper constructive bounds on $n(k, 5)$, for $k \in \{13, 14, 17, 18, \ldots\}$ that improve the best so far known.

Keywords: Regular graphs, girth, cage, amalgam.

1 Introduction

All considered graphs are finite and simple. Let G be a graph with vertex set V and edge set E. The *girth* of a graph G is the size of a shortest cycle. The set of vertices adjacent to a vertex $v \in V$ is denoted $N(v)$, and the *degree* of v is the cardinality of $N(v)$. A graph is called *k-regular* if all its vertices have the same degree k, and *bi-regular or (k_1, k_2)-regular* if all its vertices have either degree k_1 or k_2. A (k, g)-graph is a k-regular graph of girth g and a (k, g)-cage is a (k, g)-graph with the fewest possible number of vertices; the order of a (k, g)-cage is denoted by $n(k, g)$. The lower bound on the number of vertices of a (k, g)-graph is denoted by $n_0(k, g)$, and it is given as:

$$n_0(k,g) = \begin{cases} 1 + k + k(k-1) + \cdots + k(k-1)^{(g-3)/2} & \text{if } g \text{ is odd;} \\ 2(1 + (k-1) + \cdots + (k-1)^{g/2-1}) & \text{if } g \text{ is even.} \end{cases}$$

There has been intense work related with *the cage problem*, focussed on constructing the smallest (k, g)-graphs (for a complete survey of this topic see [5]). In this work, we deal with *the cage problem* for $g = 5$; in this case $n_0(k, 5) = 1 + k^2$. This bound is only attained for $k = 2, 3, 7$ and perhaps for $k = 57$ ([8]). The other known cages are for $k = 4, 5, 6$ (see [10,11,15]). When $k \geq 8$, the cage problem has become so difficult that authors resort to constructing a $(k, 5)$-graph with fewer vertices than the one constructed before. Jørgensen [9] establishes that $n(k, 5) \leq 2(q - 1)(k - 2)$ for every odd prime power $q \geq 13$ and $k \leq q + 3$. Abreu et al. ([2]) prove that $n(k, 5) \leq 2(qk - 3q - 1)$ for any prime $q \geq 13$ and $k \leq q + 3$, improving Jørgensen's bound except for $k = q + 3$, where both coincide.

2 Results

In [6] Funk uses a technique that consists in constructing regular graphs of girth at least five and performing some operations of amalgams and reduc-

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tions of the (bipartite) Levi Graph of an elliptic semiplane, either of type C or L (see [1,2,6]). In this work we also use the techniques given in [1,2] to obtain new better upper bounds on $n(k,5)$, this time amalgamating not only regular graphs but also bi-regular graphs. This way, new small $(k,5)$-graphs are constructed for $k \in \{13, 14, 17, 18 \dots 33\}$. The orders of these new graphs (thus, upper bounds for $n(k,5)$) appear in the last column of Table 1 as $New\ rec(k,5)$, which also shows for $8 \le k \le 33$ the value of $rec(k,5)$ (the smallest currently known order of a k-regular graph of girth 5, according to the notation in [5,6]).

Each of the bounds on $n(k,g)$ in Table 1 needs the construction of four *suitable* graphs. For values $k \in \{34, \dots, 255\}$, we return to the technique of Funk [6] that demands only two *suitable* graphs. In Table 2 we compare the new bounds and the existing ones for $k \in \{34, \dots, 57\}$.

It is now our interest to establish general bounds. We do that by considering the connection between difference sets and graphs. Within this framework, different families of graphs obtained from distinct families of difference sets are amalgamated.

The first difference set is obtained by Singer ([14]) in 1938. Jørgensen in [9] considers this difference set in [14] and establishes that $n(q+\lfloor\sqrt{q-1}/4\rfloor, 5) \le 2(q^2-1)$ for an odd prime power q. Also considering the Singer's difference set, we obtain the next result.

Theorem 2.1 *Let $q \ge 43$ be an even prime power such that $p = \frac{\sqrt{2q-5}-1}{2}$ is a power of a prime. Then*

$$n\Big(q + \frac{\sqrt{2q-5}-1}{2}, 5\Big) \le 2q^2 - q - 1.$$

An affine analogue of Singer's result [14] was proved in 1942 by Bose [3]. Considering Bose's difference set in [3], we establish the next general result.

Theorem 2.2 *The following bounds hold:*

(*i*) *Let $q \ge 97$ be an even prime power such that $p = \sqrt{\frac{q+1}{2}}$ is a power of a prime. Then*

$$n(q+p+2, 5) \le 2(q^2-1).$$

(*ii*) *Let $q \ge 321$ be an even prime power, and let $p \ge 9$ be the highest power of a prime such that $p \le \frac{\sqrt{q+3}}{2}$. Then,*

$$n(q+p+2, 5) \le 2(q^2-1).$$

k	$rec(k,5)$	Due to	Reference	*New* $rec(k,5)$
8	80	Jørgensen, Royle	[9,12]	
9	96	Jørgensen	[9]	
10	124	Exoo	[4]	
11	154	Exoo	[4]	
12	203	Exoo	[4]	
13	230	Exoo	[4]	*226*
14	284	Abreu et al.	[2]	*280*
15	310	Abreu et al.	[2]	
16	336	Jørgensen	[9]	
17	448	Schwenk	[13]	*436*
18	480	Schwenk	[13]	*468*
19	512	Schwenk	[13]	*500*
20	572	Abreu et al.	[2]	*564*
21	682	Abreu et al.	[2]	*666*
22	720	Jørgensen	[9]	*704*
23	880	Funk	[6]	*874*
24	924	Funk	[6]	*920*
25	968	Funk	[6]	*960*
26	1012	Funk	[6]	*1010*
27	1056	Funk	[6]	*1054*
28	1200	Funk	[6]	*1192*
29	1248	Funk	[6]	*1240*
30	1404	Funk	[6]	*1392*
31	1456	Funk	[6]	*1444*
32	1680	Jørgensen	[9]	*1608*
33	1856	Funk	[6]	*1664*

Table 1
Current and new values of $rec(k,5)$ for $8 \leq k \leq 33$.

Finally, thanks to Ganley ([7]) direct product difference set, we obtain another general upper bound.

Theorem 2.3 *Let $q \geq 41$ be an even prime power such that $\frac{1+\sqrt{2q-1}}{4}$ is a power of a prime. Then*

$$n\Big(q + \frac{\sqrt{2q-1}+3}{2},\ 5\Big) \leq 2q^2 - 1.$$

k	$rec(k,5)$	Due to	Reference	*New* $rec(k,5)$
34	1920	Jørgensen	[9]	*1800*
35	1984	Funk	[6]	*1860*
36	2048	Funk	[6]	*1920*
37	2514	Abreu et al	[2]	*2048*
38	2588	Abreu et al	[2]	*2448*
39	2662	Abreu et al	[2]	*2520*
40	2736	Jørgensen	[9]	*2592*
41	3114	Abreu et al	[2]	*2664*
42	3196	Abreu et al	[2]	*2736*
43	3278	Abreu et al	[2]	*3040*
44	3360	Jørgensen	[9]	*3120*
45	3610	Abreu et al	[2]	*3200*
46	3696	Jørgensen	[9]	*3280*
47	4134	Abreu et al	[2]	*3360*
48	4228	Abreu et al	[2]	*3696*
49	4322	Abreu et al	[2]	*4140*
50	4416	Jørgensen	[9]	*4232*
51	4704	Jørgensen	[9]	*4324*
52	4800	Jørgensen	[9]	*4416*
53	5298	Abreu et al	[2]	*4608*
54	5404	Abreu et al	[2]	*4704*
55	5510	Abreu et al	[2]	*4800*
56	5616	Jørgensen	[9]	*5304*
57	6370	Abreu et al	[2]	*5408*

Table 2
Current and new values of $rec(k,5)$ for $34 \leq k \leq 57$.

3 Future work

As the reader can see, the Jørgensen general result ([9]) as well as the ones in Theorems 2.1, 2.2, 2.3 hold for *odd* prime powers q. At this moment, we are working to obtain similar results for *even* prime powers q.

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