Virtually Fibering Random Right-Angled Coxeter Groups - Extended Abstract

Gonzalo Fiz Pontiveros

BGSMath/UPC
Barcelona, Spain

Roman Glebov

Hebrew University of Jerusalem
Jerusalem, Israel

Ilan Karpas

Hebrew University of Jerusalem
Jerusalem, Israel

1 Introduction

A group K *virtually algebraically fibers* if there is a finite index subgroup K' admitting a surjective homomorphism $K' \to \mathbb{Z}$ with finitely generated kernel. This notion arises from topology: a 3-manifold M is virtually a surface bundle over a circle precisely when the fundamental group of M virtually algebraically fibers (see the result of Stallings [10]). A *Right-Angled Coxeter group* (RACG) K is a group given by a presentation of the form

$$\left\langle x_1, x_2, \ldots x_n \mid x_i^2, [x_i, x_j]^{\sigma_{ij}} \ : 1 \leq i < j \leq n \right\rangle$$

where $\sigma_{ij} \in \{0, 1\}$ for each $1 \leq i < j \leq n$. One can encode this information with a graph Γ_K whose vertices are the generators $x_1, \ldots, x_n$ and $x_i \sim x_j$ if and only if $\sigma_{ij} = 1$. Conversely given a graph G on n vertices, we will denote the corresponding RACG by $K(G)$.

Random Coxeter groups have been of heightened recent interest, see for instance Charney and Farber [4], Davis and Kahle [5], and Behrstock, Falgas-Ravry, Hagen, and Susse [1].

Recently, Jankiewicz, Norin, and Wise [8] developed a framework to show virtual fibering of a RACG using Betsvina-Brady Morse theory [3] and ultimately translated the virtual fibering problem for K into a combinatorial game on the graph Γ_K. The method was successful on many special cases and also allowed them to construct examples where Betsvina-Brady cannot be applied to find a virtual algbraic fibering.

A natural question to consider is whether this approach is successful for a 'generic' RACG, i.e., given a probability measure μ_n on the set of RACG's of rank at most n, is it true that a.a.s. as $n \to \infty$, a group sampled from μ_n virtually algebraically fibers. This question is also considered in [8], specifically they consider sampling Γ_K from the Erdős-Renyi random graph model $\mathcal{G}(n,p)$ and they prove the following result:

Theorem 1.1 (Jankiewicz-Norin-Wise) *Assume that*

$$\frac{(2\log n)^{\frac{1}{2}} + \omega(n)}{n^{\frac{1}{2}}} \leq p < 1 - \omega(n^{-2}),$$

and let G be sampled from $\mathcal{G}(n,p)$. Then, asymptotically almost surely, the associated Right-Angled Coxeter group $K(G)$ virtually algebraically fibers.

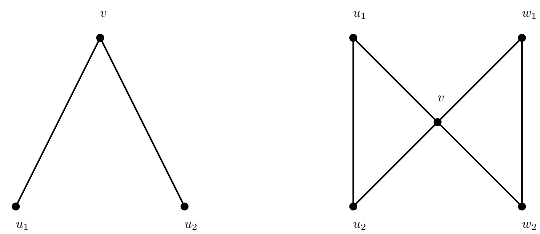
In this paper we extend this result to the smallest possible range of p, in fact we prove a hitting time type result. Namely we show that as soon as Γ_K has minimum degree 2 then a.a.s. K virtually algebraically fibers.

Theorem 1.2 *Let $G_0, G_1, \ldots, G_{\binom{n}{2}}$ denote the random graph graph process on n vertices where $G_{i+1} = G_i \cup \{e_i\}$ and e_i is picked uniformly at random from the non-edges of G_i. Let $T = \min_t \{t \ : \delta(G_t) = 2\}$, then a.a.s. the random graph process is such that $K(G_m)$ virtually algebraically fibers if and only if $T \leq m < \binom{n}{2}$. In particular for any p satisfying*

$$\frac{\log n + \log\log n + \omega(n)}{n} \leq p < 1 - \omega(n^{-2})$$

and G $\mathcal{G}(n,p)$, the random Right-Angled Coxeter group $K(G)$ virtually algebraically fibers a.a.s.

Fig. 1. A couple of toy examples.



2 The combinatorial game

In this section we follow the definitions in [8] to present the combinatorial game introduced in [8] used to construct virtual algebraic fiberings of Right-Angled Coxeter groups.

Definition 2.1 Let $G = (V, E)$ be a graph. We say that a subset $S \subset V$ is a *legal state* if both S and $V \setminus S$ are non-empty *connected subsets* of V, i.e., the corresponding induced graphs are connected and non-empty.

Definition 2.2 For each $v \in V$, a *move at* v is a set $M_v \subseteq V$ satisfying the following:

- $v \in M_v$.
- $N(v) \cap M_v = \emptyset$.

Let $\mathcal{M} = \{M_v \ : v \in V\}$ denote a set of moves.

We will identify subsets of V as elements of $\mathbb{Z}_2^V$ in the obvious way. Thus each state and each move correspond to elements of $\mathbb{Z}_2^V$ and we will think of moves acting on states via group multiplication (or addition in this case).

Definition 2.3 For a graph G, a state $S \subseteq V(G)$, and a set of moves $\mathcal{M} = \{M_v \ : v \in V\}$, the triple $(G, S, \mathcal{M})$ is a *legal system* if for any element $g \in \langle \mathcal{M} \rangle$, $g(S)$ is a legal state of G.

Theorem 2.4 ([8]) *Let $(G, S, \mathcal{M})$ be a legal system, then the RACG $K(G)$ must virtually algebraically fiber.*

To elucidate the notion of a legal system, let us look at some toy examples (see Figure 2) and ask whether each of these graphs contains a legal system.

Example 2.5 Let $G = (V, E)$ be a graph with three vertices $V = \{v, u_1, u_2\}$ and two edges $E = \{\{v, u_1\}, \{v, u_2\}\}$. We show that G has a legal system. Our initial legal state will be $S = \{u_1\}$. For our set of moves we choose

$M_v = \{v\}$ (note that this is the only possible choice for the move at v), $M_{u_1} = M_{u_2} = \{u_1, u_2\}$. Then the group generated by the moves of the graph, written as a collection of sets, is $\langle \mathcal{M} \rangle = \{\{v\}, \{u_1, u_2\}, \{v, u_1, u_2\}, \emptyset\}$. Hence, for any element $g \in \langle \mathcal{M} \rangle$, $g(S)$ is either a set of the form $\{u_i\}$ or $\{v, u_i\}$, for $i = 1, 2$, and in any case a legal state. Thus, $(G, S, \mathcal{M})$ is a legal system.

The graph in Example 2.5 is unique in the sense that it is the only graph with a vertex of degree 1 on at least 3 vertices which contains a legal system.

Next, we look at an example of a graph without a legal system. We proceed by exhaustion.

Example 2.6 Let G be the bowtie graph on 4 vertices . Assume by contradiction that $(G, S, \mathcal{M})$ is a legal system. Since v is connected to all other vertices in the graph, we must have $M_v = \{v\}$. For the same reason, v cannot belong to any other move apart from M_v. Hence, we can assume without loss of generality that $v \notin S$. Since S is a connected subset of V, we can again assume without loss of generality that $S = \{u_1\}$ or $S = \{u_1, u_2\}$.

In the latter case, $M_{w_i} = \{u_1, u_2, w_i\}$ for $i = 1, 2$, because by the definition of a move, it must be the case that $\{w_i\} \subseteq M_{w_i} \subseteq \{w_i, u_1, u_2\}$, and if u_1 or u_2 would not belong to M_{w_i}, then $M_{w_i}S$ would not be a legal state. But then the set $\{w_1, w_2\} \in \langle \mathcal{M} \rangle$, and $\{w_1, w_2\}S = \{w_1, w_2, u_1, u_2\}$ is not a legal state. In the former case, from similar consideration, it must be the case that $M_{w_i} = \{w_i, u_1\}$ for $i = 1, 2$, but then again $\{w_1, w_2\} \in \langle \mathcal{M} \rangle$, and $\{w_1, w_2\}S = \{w_1, w_2, u_1\}$ is not a legal state.

3 Quick note on method.

The first ingredient of the proof is to pick the colour classes of vertices as the moves and to choose the starting set S uniformly at random (independently of the graph). This observation allows us already get close to the threshold but not all the way: for instance an obvious obstruction is that at the target density there will be bounded vertices of degree at most C with some probability bounded away from 0 and thus with some probability bounded away from 0 these will be isolated in S.

The second ingredient then is to show that one can modify the original random selection of S and the moves to accommodate for the obstructions.

Finally, in order to prove a hitting time result, we show that any graph that *deterministically* satisfies certain pseudorandom properties must accept a legal system. The task then is to show that at the hitting time T, G_T satisfies said pseudorandom properties with high probability.

References

[1] Behrstock, J., V. Falgas-Ravry, M. F. Hagen and T. Susse, *Global structural properties of random graphs*, arXiv preprint arXiv:1505.01913 (2015).

[2] Ben-Shimon, Krivelevich and Sudakov, *On the resilience of Hamiltonicity and optimal packing of Hamilton cycles in random graphs* .

[3] Bestvina, M. and N. Brady, *Morse theory and finiteness properties of groups*, Inventiones Mathematicae (1997), pp. 445–470.

[4] Charney, R. and M. Farber, *Random groups arising as graph products*, Algebraic & Geometric Topology **12** (2012), pp. 979–995.

[5] Davis, M. W. and M. Kahle, *Random graph products of finite groups are rational duality groups*, Journal of Topology (2014).

[6] Glebov, R., *On Hamilton cycles and other spanning structures*, Ph.D. Thesis (2014).

[7] J., M., *Equitable coloring*, Amer. Math. Monthly **80** (1973), pp. 920–922.

[8] Jankiewicz, K., S. Norin and D. T. Wise, *Virtually fibering Right-Angled coxeter groups* (2017).

[9] Krivelevich, M. and B. Patkós, *Equitable coloring of random graphs*, Random Structures Algorithms **35** (2009), pp. 83–99.

[10] Stallings, J., *On fibering certain 3-manifolds*, Proc. 1961 Georgia conference on the Topology of (1961), pp. 95–100.