Graph-indexed random walks on pseudotrees

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Abstract

We investigate the average range of 1-Lipschitz mappings (graph-indexed random walks) of a given connected graph. This parameter originated in statistical physics, it is connected to the study of random graph homomorphisms and generalizes standard random walks on $\mathbb{Z}$.

Our first goal is to prove a closed-form formula for this parameter for cycle graphs. The second one is to prove two conjectures, the first by Benjamini, Häggström and Mossel and the second by Loebl, Nešetřil and Reed, for unicyclic graphs. This extends a result of Wu, Xu, and Zhu [5] who proved the aforementioned conjectures for trees.

Keywords: random walks, graph theory, graph homomorphisms, Lipschitz mappings

1 Introduction

We work with connected undirected graphs without loops and multiple edges. We call a graph *unicyclic* if it contains exactly one cycle and we call a graph *pseudotree* if it is a tree or a unicyclic graph. Equivalently, pseudotrees are graphs with at most one cycle.

Definition 1.1 A 1-*Lipschitz mapping* of a connected graph $G = (V, E)$ with a root $v_0 \in V$ is a mapping $f : V \to \mathbb{Z}$ such that $f(v_0) = 0$ and for every edge $uv \in E$ holds that $|f(u) - f(v)| \leq 1$. A *strong* 1-*Lipschitz mapping* is a 1-Lipschitz mapping where $|f(u) - f(v)| = 1$ for every edge $uv \in E$.

The set of all 1-Lipschitz mappings of a graph G is denoted by $\mathcal{L}(G)$.

Definition 1.2 The *average range* of 1-Lipschitz mappings of a graph G is defined as $\bar{\mathrm{r}}(G) := (\sum_{f \in \mathcal{L}(G)} \mathrm{r}_G(f))/|\mathcal{L}(G)|$, where the *range* $\mathrm{r}_G(f)$ of a 1-Lipschitz mapping f of G is the size of the image of f. We write $\bar{\mathrm{r}}_{\pm}(G)$ if we take the average of strong 1-Lipschitz mappings only.

Two fundamental conjectures on the average range of (strong) 1-Lipschitz mappings say that paths P_n are extremal (maximal) with regards to this parameter among all connected n-vertex graphs. The first one is by Benjamini, Häggström and Mossel and the second is by Loebl, Nešetřil and Reed.

Conjecture 1.3 *[1]* (Benjamini-Häggström-Mossel) *For any connected bipartite graph G of order n, $\bar{\mathrm{r}}_{\pm}(G) \leq \bar{\mathrm{r}}_{\pm}(P_n)$ holds.*

Conjecture 1.4 *[3]* (Loebl-Nešetřil-Reed) *For any connected graph G of order n, $\bar{\mathrm{r}}(G) \leq \bar{\mathrm{r}}(P_n)$ holds.*

In 2016, Wu, Xu, and Zhu proved that LNR and BHM conjectures hold if they are restricted to trees.

Theorem 1.5 *[5] For any tree T_n on n vertices holds that $\bar{\mathrm{r}}(T_n) \leq \bar{\mathrm{r}}(P_n)$ and $\bar{\mathrm{r}}_{\pm}(T_n) \leq \bar{\mathrm{r}}_{\pm}(P_n)$.*

Our results. Our main results are formula for cycles (Theorem 2.3) and a proof of LNR and BHM conjecture for unicyclic graphs (Theorem 3.4 and 3.5).

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[1] The first author was supported by the Charles University Grant Agency, project GA UK 1158216 and by the grant SVV-2017-260452. Both authors would like to acknowledge the support of CE-ITI (P202/12/G061 of GAČR).

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2 Formula for cycles

We will prove a formula for the average range of cycle graphs C_n (Theorem 2.3). To this end, we use lattice path enumeration, Motzkin paths, and trinomial coefficients. Before we state the main theorem of this section, we define the last two mentioned concepts.

2.1 Trinomial coefficients and Motzkin numbers

The trinomial coefficients are similar to the binomial coefficients and Pascal triangle. One can similarly define trinomial coefficients in a recursive way.

Definition 2.1 *Trinomial numbers (coefficients)*, denoted $\binom{n}{k}_2$, with $n, k \in \mathbb{Z}$, are defined as $\binom{0}{0}_2 := 1$, and

$$\binom{n+1}{k}_2 := \binom{n}{k-1}_2 + \binom{n}{k}_2 + \binom{n}{k+1}_2 \text{ for } n \geq 0,$$

where $\binom{n}{k}_2 := 0$ for $k < -n$ and $k > n$.

Central trinomial coefficients are the numbers $\binom{n}{0}_2$, where $n \in \mathbb{N}_0$. The sequence for central trinomial coefficients in OEIS is A123456.

For the proof of the formula for $\bar{\mathrm{r}}(C_n)$ we need to define the *generalized Motzkin numbers and paths*. We will further write only Motzkin numbers and Motzkin paths. For more details, we refer to the seminal paper [2].

Definition 2.2 Consider a lattice path, beginning at $(0, 0)$, ending at (n, k) and satisfying that y-coordinate of every point is non-negative. Furthermore, every two consecutive steps (i, a) and $(i + 1, b)$ must satisfy $|a - b| \leq 1$. Such lattice paths are called *Motzkin paths*. The set of all Motzkin paths ending in (n, k) is denoted by $m(n, k)$ and the cardinality of this set is denoted by $M(n, k)$. We call numbers $M(n, k)$ the Motzkin numbers.

2.2 Statement of the main theorem and lemmata

We can finally state the main theorem.

Theorem 2.3 *For any $C_n, n \geq 3$, we have $\bar{\mathrm{r}}(C_n) = (3^n + (-1)^n)/(2 \cdot \binom{n}{0}_2)$.*

Let us prove a formula for $|\mathcal{L}(C_n)|$ first.

Theorem 2.4 *For any C_n, $n \geq 3$, $|\mathcal{L}(C_n)| = \binom{n}{0}_2$.*

This theorem is a simple corollary of the following observation.

Observation 1 *There is a bijection between* 1*-Lipschitz mappings of* C_n *and the set of lattice paths starting at* $(0,0)$*, ending at* $(n,0)$*, and satisfying that for every two consecutive steps* (i,a) *and* $(i+1,b)$*,* $|a-b| \le 1$*.*

We denote by $\mathcal{L}(C_n, -d)$ the set of 1-Lipschitz mappings f of C_n satisfying $\min_{v \in V(C_n)} f(v) = -d$. In other words, $\mathcal{L}(C_n, -d)$ denotes the set of all 1-Lipschitz mappings of C_n with $-d$ as the minimum value of their images.

We need to show a bijection between $\mathcal{L}(C_n, -d)$ and the set $m(n, 2d)$. The following lemma can be deduced from Theorem 1 of Van Leeuwen's paper [4] and Lemma 1. Due to space constraints, details of this are left for a future journal version.

Lemma 2.5 *There exist a bijection between the set of Motzkin paths* $m(n, 2d)$ *to the set* $\mathcal{L}(C_n, -d)$*.*

For a technical convenience, we will define the *irregular trinomial coefficients*; see the sequence A027907 in OEIS database. The *irregular trinomial coefficients* are defined for every $n, k \in \mathbb{Z}$ as $T^*(n,k) = \binom{n}{k+n}_2$.

The following lemmata, showing the relation of Motzkin paths and irregular trinomial coefficients will be crucial for the proof of the main theorem.

Lemma 2.6 *The following identity holds for every* $n, k \in \mathbb{N}_0$*,* $k \le n$*,*

$$M(n,k) = T^*(n, n-k) - T^*(n, n-k-2). \tag{1}$$

Now we need the last lemma, concerning sums of irregular coefficients.

Lemma 2.7 *For every* $n \in \mathbb{N}_0$ *holds*

(i) $\sum_{k=0}^{n} T^*(n, 2k) = (3^n + 1)/2$ *if* n *is even, and*

(ii) $\sum_{k=1}^{n} T^*(n, 2k-1) = (3^n - 1)/2$ *if* n *is odd.*

2.3 Proof of the main theorem

Proof. [Proof of Theorem 2.3] We use the following identity for every $n \ge 3$.

$$\sum_{k=0}^{\lfloor n/2 \rfloor} (2k+1)(T^*(n, n-2k) - T^*(n-1, n-2k-2)) = \begin{cases} \sum_{k=0}^{n} T^*(n, 2k), & n \text{ even} \\ \sum_{k=1}^{n} T^*(n, 2k-1), & n \text{ odd} \end{cases}$$

This identity follows from straightforward calculations and from the observation that $T^*(n, n-k) = T^*(n, n+k)$ for every $n, k \in Z$.

For brevity, we will sketch the following calculation for n odd only.

$$\begin{aligned}\bar{\mathrm{r}}(C_n)\cdot|\mathcal{L}(C_n)| &= \sum_{k=0}^{\lfloor n/2\rfloor}(2k+1)\cdot M(n,2k) && \text{(by Lemma 2.5)}\\ &= \sum_{k=0}^{\lfloor n/2\rfloor}(2k+1)\cdot\big(T^*(n,n-2k)-T^*(n-1,n-2k-2)\big)\\ & && \text{(by Lemma 2.6)}\\ &= \sum_{k=1}^{n}T^*(n,2k-1) = (3^n-1)/2. && \text{(by Lemma 2.7)}\end{aligned}$$

Together with Theorem 2.4 taken into account we conclude the formula for $\bar{\mathrm{r}}(C_n)$. □

3 LNR and BHM conjectures for unicyclic graphs

Lemma 3.1 *The number of 1-Lipschitz mappings of unicyclic graphs with order n and cycle size k, $k \leq n$, is equal to $\binom{k}{0}_2 \cdot 3^{n-k}$.*

Observe that Lemma 3.1 implies that every two same-order unicyclic graphs with the same-length cycle have the same number of Lipschitz mappings.

It will be useful to give a name to one special subset of unicyclic graphs.

Definition 3.2 A *corolla graph* is a unicyclic graph obtained by taking a cycle graph and joining some path graphs to it by identifying their endpoints with some vertex of that cycle. Every path is joined to exactly one vertex of the cycle. And every vertex of the cycle has at most one path attached. We call these paths attached to vertices of the cycle as *rays*.

The main theorem of [5], Theorem 2.5, can be used to prove the following useful lemma.

Lemma 3.3 *For every unicyclic graph U of order n, which is not a corolla graph there exist a corolla graph R of order n, such that $\bar{\mathrm{r}}(U) \leq \bar{\mathrm{r}}(R)$.*

We can finally sketch the proof of the second main result of this paper.

Theorem 3.4 *For any unicyclic graph U of order n, $\bar{\mathrm{r}}(U) \leq \bar{\mathrm{r}}(P_n)$ holds.*

Proof.

- First, use Lemma 3.3 to transform U to R, a corolla graph with a cycle of size equal to k.

- The following can be proved: In every corolla graph with at least two leafs, one can delete an edge to a leaf vertex of a ray with the minimum order and attach this vertex to a leaf of a ray with the maximum possible order. This operation does not decrease the average range.
- After successively applying this operation enough times we get a *tadpole graph* $T_{n,k}$, a cycle of size k with a path of size $n-k$ attached by a bridge.
- One can further prove that for every $k \geq 4$, $\bar{\mathrm{r}}(T_{n,k}) \leq \bar{\mathrm{r}}(T_{n,k-1})$ holds. Finally, it holds that $\bar{\mathrm{r}}(T_{n,3}) \leq \bar{\mathrm{r}}(P_n)$. That completes the proof.

□

We note that our proof works also for strong 1-Lipschitz mappings and hence the following holds as well.

Theorem 3.5 *For any bipartite unicyclic graph U of order n, $\bar{\mathrm{r}}_{\pm}(U) \leq \bar{\mathrm{r}}_{\pm}(P_n)$.*

4 Concluding remarks

We proved a closed-form formula for the average range of 1-Lipschitz mappings of cycles using an analysis of lattice paths, trinomial coefficients, and Motzkin paths. Furthermore, we extended the result on trees from [5]. We proved that LNR and BHM conjectures hold if restricted to pseudotrees. A natural next candidate to extend these conjectures seems to be the class of *cactus graphs*.

References

[1] Itai Benjamini, Olle Häggström, and Elchanan Mossel. On random graph homomorphisms into Z. *Journal of Combinatorial Theory, Series B*, 78(1):86–114, 2000.

[2] Robert Donaghey and Louis W Shapiro. Motzkin numbers. *Journal of Combinatorial Theory, Series A*, 23(3):291–301, 1977.

[3] Martin Loebl, Jaroslav Nešetřil, and Bruce Reed. A note on random homomorphism from arbitrary graphs to Z. *Discrete mathematics*, 273(1):173–181, 2003.

[4] Marc AA Van Leeuwen. Some simple bijections involving lattice walks and ballot sequences. *arXiv preprint arXiv:1010.4847*, 2010.

[5] Yaokun Wu, Zeying Xu, and Yinfeng Zhu. Average range of Lipschitz functions on trees. *Moscow Journal of Combinatorics and Number Theory*, 1(6):96–116, 2016.