Transversals, domination and the LS category on surfaces[S]

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The *Lusternik-Schnirelmann* or *LS category* of a topological space X, $cat(X) = n$, is the least integer n such that there exists an open covering $U_1, \dots U_{n+1}$ of X with each U_i contractible to a point *in the space* X. If no such integer exists, $cat(X) = \infty$. It is known that $cat(S^2) = 2$ and for any other closed surface X, $cat(X) = 3$, [2].

A *hypergraph* $H = (V, E)$ with *vertex* finite set V is a family $E = \{E_1, \dots, E_m\}$ of subsets of V (the *hyperedges* of H) so that $E \neq \emptyset$ and $V = \cup_{i=1}^{m} E_i$. Two vertices are *adjacent* if they lie in a hyperedge. A hypergraph is said to be k-*uniform* if all its hyperedges are of size k. A subset of vertices is said to be a *transversal* of H if it meets every hyperedge in E. The *transversal number* $\tau(H)$ is the minimum cardinality of a transversal of H. A set of vertices $D \subset V$ is *dominating* in H if each vertex of $V - D$ is adjacent to some vertex in S. The *domination number* of H, denoted $\gamma(H)$, is defined to be the cardinality of a minimum dominating set of H. Every triangulation T on any surface (possibly with non-empty boundary) can be considered a 3-uniform hypergraph and has domination number $\gamma(T) \leq |V|/3$. See [1, 3].

Considering a triangulation of a surface S as a graph $G = (V_G, E_G)$, the *contraction* of an edge of G consists of the identification of its vertices (removing multiple edges if they appear). An edge is *contractible* if the resulting graph is still a triangulation of S. The inverse operation of the edge contraction is called *vertex splitting.* A triangulation G is *irreducible* if it has no contractible edges. It is immediate that for any irreducible triangulation G of a closed surface S one has $cat(S) \leq \tau(G)$. Moreover, the transversal number may increase under splitting vertices. This way the difference $\tau(G) - cat(S)$ can be arbitrarily large.

Problem Find a function α of the genus $g = g(S)$ of the closed surface S such that $cat(S) \leq \tau(G) \leq cat(S) + \alpha(g)$ for any irreducible triangulation G of S. For all the irreducible triangulations G of the torus S_1, $\alpha(g(S_1)) = g(S_1)$ and $3 = cat(S_1) \leq \tau(G) \leq cat(S_1) + g(S_1) = 4$.

New parameters for the class of surfaces are introduced in this work. The *domination number* and the *transversal number* of a surface S, are defined as

$\gamma(S) = max\{\gamma(G) : G \text{ is an irreducible triangulation of } S\}$,

$\tau(S) = max\{\tau(G) : G \text{ is an irreducible triangulation of } S\}$, respectively.

Examples: $\gamma(S_0) = 1$; $\gamma(S_1) = 2$; $\gamma(N_1) = 1$; $\gamma(N_2) = 2$.

Theorem 1. *For any closed surface S with Euler-Poincaré characteristic $\chi(S) \leq 1$,*

$$\gamma(G) \leq \frac{22 - 13\chi(S)}{3}$$

[1] Chávez, M. J., Quintero, A. and Villar-Liñán, M. T. *Triangulated surfaces with a given domination number.* X Encuentro Andaluz de Matematica Discreta, La Línea de la Concepción, (Cádiz), 2017.

[2] Cornea, O. Lupton, G., Oprea, J. Tanré, D. *Lusternik-Schnirelmann Category.* Mathematical Surveys and Monographs, vol. 103, Amer. Math. Soc. (2003).

[3] Furuya, M., and Matsumoto, N. A note on the domination number of triangulations, *J. of Graph Theory*, (2014), 83-85.

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