Pascal automata and matroids

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Let G be a finite set with a binary operator $*$, called product, such that $(G, *)$ forms an abelian group, and let h denote its exponent. Given N and s two natural numbers such that $2 \leq s \leq N$, we define the following one–dimensional finite cellular automaton, or CA (see [3]): The initial cell set (first row which corresponds with time $t = 0$) is an ordered linear array of N cells which are denoted by $(0, i)$, $1 \leq i \leq N$. The evolution of the CA throughout the time gives rise to linear arrays (rows) of different length, whose cells are designed as (t, i), with $(t, i) \in \mathbb{N}^2$, where t refers to the time period and i is the cell position in the array. The state set of the CA is the group G and, in general, g_i^t denotes the state of the i-th cell of the row obtained at time t. The neighborhood of the cell (t, i) consists of s cells belonging to the $(t-1)$-th row, according to the following CA's local evolution rule:

$$g_i^t = g_i^{t-1} * g_{i+1}^{t-1} * \ldots * g_{i+s-1}^{t-1}. \tag{1}$$

Note that this rule involves $s+1$ elements of G: s states of adjacent cells in the same row and another state of a cell in the next row or time period, such that this state is the product of those ones. The underlying abelian group structure of G implies that knowledge of any s elements of this configuration, called a *cell basic configuration*, uniquely determines the state of the remaining cell.

Then, the orbit of the CA, starting from the states of the N initial cells, is a two–dimensional arrangement of elements g_i^t of G, ordered in rows and columns, called a *triangle*, and it is straightforward to see that it is uniquely determined by the values of N and s, and the states of the cells $(0, i)$, with $1 \leq i \leq N$.

The set made up of all triangles is denoted by $\nabla_N(G, s)$. Then, $\nabla_N(G, s)$ should be seen under two perspectives: On one hand, it collects all triangles according to the above definition. On the other hand, $\nabla_N(G, s)$ can be considered as an ordered set of empty cells, which are connected by the cell basic configuration. We would like to stress that it is possible to unambiguously identify triangles of $\nabla_N(G, s)$ from the states of other cells different to the collection of N entries of the first row, following the above–mentioned property of cell basic configurations, stated in Equation 1. In particular, we are interesting in characterizing the cell sets whose states determine an unique triangle of $\nabla_N(G, s)$.

A cell c *is generated* by a cell subset A of $\nabla_N(G, s)$, or A *generates* c, if the states of the cells of A uniquely determine the state of c applying the CA's local evolution rule of cell basic configurations.

A cell subset A of $\nabla_N(G, s)$ is *independent* if, for any assignment of states to its cells, there exists at least a triangle of $\nabla_N(G, s)$ containing these states for the cells of A. On the contrary, a cell subset A of $\nabla_N(G, s)$ is *dependent* if there exists an assignment of states to its cells for which there is no triangle of $\nabla_N(G, s)$ including these states for the cells of A.

It is straightforward to check that none cell of an independent set can be generated by its remaining cells. Conversely, any cell of a dependent set is generated by its remaining cells.

Given an independent set A of $\nabla_N(G, s)$, let $\mathcal{G}_A$ be the set of cells generated by A. Then, a cell independent set A of $\nabla_N(G, s)$ is a *generating set* of $\nabla_N(G, s)$ if $\nabla_N(G, s) = A \cup \mathcal{G}_A$. Thus, one of our aims in this paper is to characterize all generating sets of $\nabla_N(G, s)$.

Let $\mathcal{M}(\nabla_N(G, s))$ be the ordered pair $(\nabla_N(G, s), \mathcal{I})$ where $\mathcal{I}$ is the collection of independent sets of $\nabla_N(G, s)$. Then it can be proved that $\mathcal{M}(\nabla_N(G, s))$ is a matroid that we call *Pascal matroid*. Depending on the values of parameters s, N and h, the obtained matroids have a great variety of structures, ranging from graphic to ternary [2].

The main result of this work is that operating with the elements of the matroid, it is simple to check for any subset of cells of $\nabla_N(G, s)$ whether or not they are independent, in the sense that one of them can be generated from the others. Some other authors [1] have partially solved this problem by using a much more complicated construction.

References

[1] A. Barbé, F. von Haeseler, The Pascal matroid as a home for generating sets of cellular automata configurations defined by quasigroups. Theor. Comput. Sci. 325 (2004) 171–214.

[2] J.G. Oxley, Matroid Theory. Oxford Science Publications, The Clarendon Press, Oxford University Press, New York, 1992.

[3] Wolfram, S. A New Kind of Science. Champaign, IL, Wolfram Media, 2002.

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