Can labelled vertices make the graph euclidean realizable?

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There exists a broad literature on realization of graphs in the euclidean space (euclidean graphs), in particular unit–distance graphs (see, for instance [1, 2]). On one hand, given a graph with positive edge weights, the aim of realization problems is to determine whether it is possible to map its vertices to different points in $\mathbb{R}^d$ and its edges to segments (without crossings) joining the corresponding points in such a way that the length of each segment is just its weight. On the other hand, it is a natural approach to consider weighted or labelled vertices, as, for example, in the bandwidth problem or in graceful labelings of graphs; nevertheless, we have not found references about graph realization problems in the case of edge weights coming from vertex weights or labels. Thus, we address realization problems of graphs with labelled vertices.

Let $G(V,E)$ be a graph and $f : V(G) \longrightarrow \mathbb{R}$ be a vertex labeling function. On one hand, a weighting on the edges of G is defined as $\widehat{f}(e) = |f(u) - f(v)|$, for every edge $e = \{u,v\}$ of E. On the other hand, f induces directions to edges whether each edge is oriented from its vertex with smallest label to its vertex with greatest label. Thus, f makes G an edge–weighted digraph. Note that f is a proper vertex coloring of G if adjacent vertices have different labels.

It is said that the pair (G,f) is *realizable* in $\mathbb{R}^d$ if G is realizable for edges weighted by $\widehat{f}$. Obviously, a necessary condition for realizability is that f is a proper vertex coloring, hence, from now on, we deal with proper vertex colorings. A graph G is *realizable* in $\mathbb{R}^d$ if there exists a vertex labeling function f such that (G,f) is realizable in $\mathbb{R}^d$. Thus, we can defined the *realizability index* of G, denoted $ri(G)$, as the minimum dimension d such that G is realizable in $\mathbb{R}^d$, and we say that $ri(G) = +\infty$ if G is not realizable in any $\mathbb{R}^d$.

We define a *monotonous* path of (G,f) as any directed path of the digraph induced by f, that joins two adjacent vertices.

Theorem 1 *Given a graph G and a proper vertex coloring f, the following statements are equivalent:*

1. *(G,f) is realizable in $\mathbb{R}^d$.*
2. *There is no pair of adjacent vertices u and v in G such that $d_G(u,v) = d_{G-\{u,v\}}(u,v)$, where distances are considered on weighted graphs.*
3. *There is no monotonous path for (G,f).*

The following facts are direct consequences of the second or third statement of Theorem 1.

Corollary 2 *If a graph is realizable for some coloring, then it is triangle–free.*

Corollary 3 *If the chromatic number of a graph G is smaller than its girth then G is realizable.*

Theorem 4 *It is possible to check in polynomial time whether a pair (G,f) is realizable or not.*

If we focus on the family of bipartite graphs (of particular interest since they are unit–distance graphs), the following result can be stated:

Theorem 5 *The realizability index of any bipartite graph is at most 3. Furthermore, for any d there exists a bipartite graph G such that (G,f) is not realizable in $\mathbb{R}^d$ (where f represents any bicoloring of G).*

In this context, we have initiated the study of some questions, among them:

1. If G is realizable find $ri(G)$.
2. Characterize the graphs with low realizability index.
3. Find the realizability index for bipartite graphs.

References

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