Curvature criteria to fit curves to discrete data

Lyuba Alboul *,[a] Gilberto Echeverria [a] Marcos Rodrigues [a]
[a] *Sheffield Hallam University, Sheffield S1 1WB, UK*
http:www.shu.ac.uk/scis/artificial_intelligence

Abstract

Several geometric criteria to fit a polygonal closed curve to discrete two-dimensional data are considered and analysed. Most of these criteria are related to the concept of curvature, for example, one criterion is minimisation of total absolute curvature. On the basis of these criteria algorithms to construct a polygonal curve which is optimal with respect to a specific criterion, are designed.

Key words: Curve reconstruction, total absolute curvature, polygonisation

1. Introduction

The work presented in this communication is dedicated to the following problem: *Given discrete point data set S in 2D, find a curve that spans all these points and best satisfies a specific criterion.* We call this problem the *Curve modelling problem*, and in what follows we abbreviate it as the *Curvmod* problem. This problem occurs in particular forms in various applications. For example, an important problem in Computer Graphics is to find a planar curve that is shaped in the way as you want it to be shaped [8]. A well–known form of the *Curvmod* problem is the *Curve Reconstruction Problem* which is concerned with finding a piecewise linear approximation to a curve from a set of given sample points [6]. An algorithm for curve reconstruction should preserve the order of the points sampled from the curve, *i.e.*, the points are connected only if they are adjacent along the curve. Applications of this problem can be found in many areas, for example, detecting boundaries in Image Processing, determining patterns in Computer Vision, and reconstruction of topographic objects from aerial images.
Another form of the *Curvmod* problem is related to polygonisation of a given planar point set S. Studying the set of polygonisations of S is an active area of research in computational geometry. For example, one of the polygonisation problems is to determine a polygonisation (or polygonisations) with the minimum number of *reflex* vertices, as the number of reflex vertices quantifies in a combinatorial sense the degree to which the point set S is in convex position [5].
We restrict ourselves to polygonal curves. There are various algorithms for curve reconstructing, which give good results if the sampling of data points is dense enough [4,6]. We look at the problem from another point of view. We omit any additional information on the data points, and attempt to determine what 'reasonable' curves can be constructed for the given data and what characteristics these curves possess and how these characteristics distinguish one curve from another. In this setting the *Curvmod* problem can be viewed as an optimisation problem: a 'reasonable' curve can be defined as an *optimal curve* with respect to a certain criterion.
We introduce several geometrical criteria, mostly related to the concept of curvature, as curvatures truly describe the shape of objects. Research on discrete curvatures (*i.e.*, curvatures that can be computed on a set of points) is of growing interest in geometric modelling (some overview is given in [2]). An important measurement of an object is its size, some criteria related to this measurement are also considered.

* Corresponding author
Email address: L.Alboul@shu.ac.uk (Lyuba Alboul).

2. Preliminaries. Notions related to the concept of curvature

2.1. *Problem definition*

The definition of the curvature of a curve is in general given under the assumption that the curve is of C^2–class, and thus posseses two continuous derivatives. However, the notion of curvature can be generalised to the class of curves which possess continuous second derivatives except at a finite number of points where a jump discontinuity in the first derivative can occur. We can introduce the *exterior angle* $\alpha(s)$ formed by the right and left tangents at a point of discontinuity s [9]. In the case of a polygonal curve points of discontinuity are its vertices. In this paper we discuss only curves, which are continuous images of the circle S^1 into the plane, *i.e.*, *closed polygons*. Let us denote with $\alpha(v_i)$ the exterior angle at the vertex v_i of a curve. Then expression

$$\omega = \sum_{i=1}^{n} \alpha(v_i) \qquad (1)$$

represents the total curvature of a closed polygonal curve. If this curve is the boundary of a closed simple polygon, ω is always equal to 2π. Its absolute value $|\omega|$, or *absolute total curvature* is more informative. An important fact is that $|\omega|$ reaches its minimal value 2π on convex curves (polygons). With respect to the concept of *total absolute curvature* we can formulate the following problem:

Problem 1. Given discrete two–dimensional data. Among all closed polygonal curves that span these data find a curve (or curves) with the minimal total absolute curvature.

Motivation to study polygonal curves of *minimal total absolute curvature* is related to the concept of Tight submanifolds [7]. One of the main properties of one– and two–dimensional tight submanifolds in $\mathbb{R}^3$ is that they possess the minimal total absolute curvature. For non–regular curves and surfaces such as polygonal curves and polyhedral surfaces the concept of curvature is also determined, and surface triangulations of minimal total absolute curvature (MTAC) were introduced. MTAC triangulation coincides with the convex triangulation of the data if the data are in convex position, but the properties of triangulations with MTAC for non-convex data are still mostly unknown (see [3] for a review). As surfaces are much more complex objects than curves, the study of the one–dimensional version of the MTAC triangulation provides useful insights for research on triangulations. Besides this, the study of polygonal curves with respect to their total absolute curvature is also interesting for its own sake, as the total absolute curvature is related to the global properties of the curve and might be used to characterise diverse curve profiles. We are interested to determine *to what extent a polygonal curve is represented by its total absolute curvature.*

2.2. *Some simple formulae and examples*

The total curvature and total absolute curvature coincide for a closed convex polygonal curve and both equal to 2π. Therefore for a non–convex curve the excess in the total absolute curvature with respect to the curvature of the convex curve can be used as a measure of deviation from the convex curve.

Given the data, we take as a *convex curve of reference* the boundary of the convex hull of the data, (referred to as the *CB–curve*). The statement of Problem 1 allows self–intersecting curves, but in this communication we presume that polygonal curves represent boundaries of simple polygons, *i.e.*, are not self–intersecting.

A simple polygonal curve L, that spans the data, is represented by its vertices V_1, V_2, ..., V_{n-1}, V_n. V_iV_j, where $j = i + 1$, is a line segment, and its length is denoted with l_{ij}, $l_{ij} = l_{ji}$. A *star* of a vertex V_i is the union of the vertex and its two adjacent line segments $V_{i-1}V_i$ and V_iV_{i+1}.

We designate the vertices that lie on *CB–curve* with V^i_{CB}, and with $\gamma_i, i = 1, ..., l$ the exterior angles at these vertices with respect to the *CB–curve*. It is clear that $\sum \gamma_i = 2\pi$.

Let us denote with V^j_{conv} convex vertices of L and with $\alpha_j, j = 1, ..., m$ the exterior angles at these vertices with respect to L, and with V^k_{reflex} – reflex vertices of L and with $\beta_k, i = 1, ..., p$, the corresponding exterior angles at these vertices with respect to L; $m + p = n$.

For a simple closed polygon the following equality holds:

$$\sum \alpha_j - \sum \beta_k = 2\pi \qquad (2)$$

As $|\omega| = \sum \alpha_j + \sum \beta_k$, the total absolute curvature of a polygonal closed simple curve can be represented as

$$|\omega| = 2\pi + 2\sum_k \beta_k \tag{3}$$

The set of convex vertices V^j_{conv} can be further split into three disjoint subsets, namely, the subset $V^{j_1}_{conv-CB}$ of vertices that lie on the CB–curve and such that their stars belong to the CB–curve; the subset $V^{j_2}_{conv-int}$ of vertices that are convex but do not belong to the CB–curve; and the subset $V^{j_3}_{conv-corn}$ of vertices that lie on the CB–curve but their stars do not belong to the CB–curve. Vertices of the last type are called *corner vertices* as at these vertices the curve is deviated from the CB–curve. The corresponding exterior angles are denoted as α_{j_1}, α_{j_2} and α_{j_3}. Obviously, any part of the curve that is deviated from the CB–curve starts and ends at the neighbouring vertices of $V^{j_3}_{conv-corn}$, since our curve has no self–intersections. Each α_{j_3} is equal to $\gamma_{j_3} + \alpha(CB)_{j_3}$; where by $\alpha(CB)_{j_3}$ we denote the angle at a corner vertex of the curve L with respect to the CB–curve, or in other words, the angle between the edge of L that has this corner vertex as one of the end-vertices and the (imaginary) edge of the CB-curve, that would have the same corner vertex as one of the end-vertices. We call such an angle a *deviating angle*. After some computation, we obtain the following expression:

$$\sum(\alpha_{j_2} + \alpha(CB)_{j_3}) = \sum \beta_k \tag{4}$$

Therefore, to find the curve (or curves) of minimal total absolute curvature among all curves that span the given data, it is sufficient to minimise either the sum of exterior angles at reflex vertices or the sum of exterior angles at the internal convex vertices and the deviating angles.
From formula 4 it follows that if the curve does not have internal convex vertices then the amount of deviation of the curve from the convex curve CB is concentrated in deviating angles. Each deviated part L_d, $d = 1, ..., f$ contains in its turn some convex internal vertices $V^{j^d_2}_{conv_d}$ (its number may be equal to zero) and reflex vertices $V^{k^d}_{reflex_d}$
Equality 4 holds for each deviated part:

$$\sum(\alpha_{j^d_2} + \alpha(CB)_{j^d_3}) = \sum \beta_{k^d} \tag{5}$$

Let us define a convex region R_{conv} as a part $V_i, V_{i+1}, ..., V_{i+j}$ of a curve L that satisfies the condition that each vertex that belongs to R_{conv} is convex, but vertices V_{i-1} and V_{i+j+1} are reflex.

A concave region $R_{concave}$ is defined analogously. The following statement is true:
Suppose we construct a curve L_1 which has g convex regions and h concave ones. If we manage to add new vertices in such a way that the obtained regions are preserved, than a new curve L_2 will possess the same total absolute curvature as L_1.

The *spherical image* of a curve visualises the concept of *total absolute curvature*. The spherical image for a polygonal curve is constructed by means of outwards units normals to the line segments of the curve, all of them are 'translated' to the same origin. The ends of the unit normals of a planar curve will lie on the unit circle. Let us suppose that we walk around the boundary of a polygon, for example, in anticlockwise direction starting from vertex V_1, passing through all the vertices according to their order until we arrive again at the vertex V_1. To this walk a corresponding walk on the circle is generated, some of its parts are walked several time for a non–convex curve.
The spherical image provides a vertex classification of the curve as for a reflex vertex the direction will be opposite to the chosen one. The spherical images can be put in one–to–one correspondence for two curves of the same data set if the numbers of concavities/convexities and corresponding 'incorporated' curvatures for the both curves are the same. We say in this case that two curves are *curvature identical*. Examples of two *curvature identical* curves and their corresponding spherical images (schematically depicted), are given in Fig. 1, Fig. 2, Fig. 3 and Fig. 4.

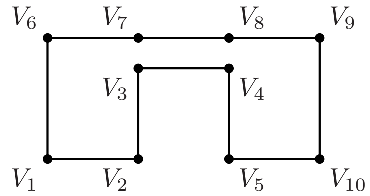


Fig. 1. Ten–points data set. Curve 1

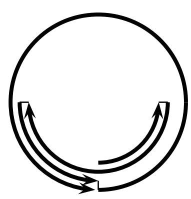


Fig. 2. A spherical image of Curve one

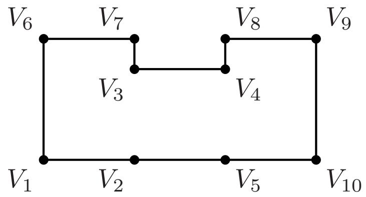


Fig. 3. Ten–points data set. Curve 2

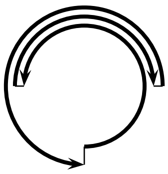


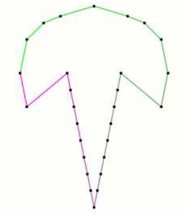
Fig. 4. A spherical image of Curve two

3. Short overview of research

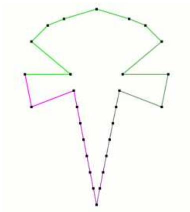
In order to construct 'reasonable' curves we consider the following criteria:

(i) minimisation of total absolute curvature;
(ii) minimisation of the length of a curve;
(iii) minimisation of the average curvature of a curve, *i.e.*, $|\omega|/\sum l_{ij}$;
(iv) minimisation of the total absolute curvature of a curve with only one 'deviation';
(v) minimisation of various weighted curvatures, such as $\sum_i(|\alpha(v_i)|(l_{(i-1)i} + l_{i(i+1)})$, $\sum_i |\alpha(v_i)|l_{(i-1)i}l_{i(i+1)}$ and some other.

Several algorithms are designed in order to obtain optimal curves with respect to these criteria. The objective of this research is to study how various criteria influence the shape of a curve. Therefore, we experimented with algorithms as well, at this moment we are not really interested in their costs. For certain criteria more than one algorithm is designed, using local or global approaches. See Fig. 5 for examples of outputs of two different algorithms to construct a curve of minimal total absolute curvature. We tested the algorithms on many data sets, among them data from curves with sharp corners, and onions with several layers when the minimum distance between the layers is greater than a certain value ϵ. In order to compare curves obtained by different criteria various geometric shape parameters are computed. The research is still in its development. One of the goals is to optimise some of obtained algorithms. All details and illustrative examples are given in the full version of the paper [1].



(i)



(ii)

Fig. 5. Outputs of two different algorithms for the same input data. The curve in (i) is the curve of minimal total absolute curvatue

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