On relative isodiametric inequalities

Cerdán, A. [a], Miori, C. [a] and Segura Gomis, S. [a]

[a] *Departamento de Análisis Matemático, Universidad de Alicante, Campus de San Vicente del Raspeig, E-03080-Alicante, Spain*

Abstract

We consider subdivisions of convex bodies G in two subsets E and $G \setminus E$. We obtain several inequalities comparing the relative volume 1) with the minimum relative diameter and 2) with the maximum relative diameter. In the second case we obtain the best upper estimate only for subdivisions determined by straight lines in planar sets.

Key words: Relative geometric inequalities, relative isodiametric inequalities, area, diameter

1. Introduction

Relative geometric inequalities are inequalities in which we compare relative geometric measures, i.e., functionals that give geometric information of the subsets (E and $G \backslash E$) determined by the subdivision of an original set G.

The first relative geometric inequalities that appeared in the literature were the so called relative isoperimetric inequalities. These inequalities compare the relative area with the relative perimeter:

If G is an open convex set in the Euclidean plane $\mathbb{R}^2$ and E is a subset of G with non-empty interior and rectificable boundary such that both E and its complement $G \backslash E$ are connected, we define:

- the *relative boundary* of E as $\partial E \cap G$.

- the *relative perimeter* of E, $P(E, G)$, as the length of the relative boundary, and

- the *relative area* of E as:

$$A(E, G) = min\{A(E), A(G \backslash E)\}.$$

With the above assumptions we say that a relative isoperimetric inequality is an inequality of the type:

$$\frac{A(E, G)}{P(E, G)^{\alpha}} \leq C,$$

where C and α are positive numbers.

Email addresses: aacs@alu.ua.es (Cerdán, A.), cm4@alu.ua.es (Miori, C.), Salvador.Segura@ua.es (Segura Gomis, S.).

Many results have been obtained about relative isoperimetric inequalities (see for instance [2],[7]).

There are also results comparing the relative perimeter with other geometric magnitudes different from the relative area. For results comparing the relative perimeter with the relative diameter and the relative inradius see [4].

In this paper we want to study *relative isodiametric inequalities*, in which we compare the relative volume with the relative diameters of a subset of a convex body. First we need to define these notions:

Let $G \subset \mathbb{R}^n$ be a convex body and $E \subset G$ a subset of G such that E as well as $G \setminus E$ are connected and have non–empty interior. Let $D(.)$ be the diameter functional.

(i) the *relative volume* is the minimum between the volume of E and the volume of its complement,

$$V(E, G) = min\{V(E), V(G \setminus E)\},$$

(ii) the *minimum relative diameter* is the minimum between the diameter of E and the diameter of its complement,

$$d_m(E, G) = min\{D(E), D(G \setminus E)\},$$

and

(iii) the *maximum relative diameter* is the maximum between the diameter of E and the diameter of its complement,

$$d_M(E, G) = max\{D(E), D(G \setminus E)\}.$$

Relative isodiametric inequalities are those that give either an upper or a lower estimate of the ratios:

$$\frac{V(E,G)}{d_m(E,G)^n} \text{ or } \frac{V(E,G)}{d_M(E,G)^n}.$$

We compare the relative volume with the n-power of the relative diameters (n is the dimension of the ambient space) because as this ratio is invariant under dilatations, we obtain geometric information about the subdivision: the estimates do not depend on the size of the bodies but only on their shapes.

We are interested not only in obtaining relative isodiametric inequalities, but also in determining those sets (called maximizers or minimizers) for which the equality sign is attained.

2. Relative isodiametric inequalities concerning the relative area and the minimum relative diameter of a subset of a convex body

The aim of this section is to maximize and minimize the ratio between the relative area and the n-power of the minimum relative diameter of a subset E of G.

We begin minimizing the given ratio, and in this case we have to consider two different cases. First, we are going to study general subdivisions of G and later we are going to consider the special case in which the subdivision is obtained by a hyperplane cut.

Theorem 1 *Let G be an open convex body and E a subset of G such that E and $G \setminus E$ are connected. Then,*

$$\frac{V(E,G)}{d_m(E,G)^n} \geq 0.$$

PROOF. For any convex body G we can consider a sequence of hypersurfaces $\{S_i\}_{i=1}^{\infty}$ as close as we want to the boundary of G, such that both ends of the diameter of G belong to S_i $\forall i \in \mathbb{N}$, and the regions E_i bounded by S_i have volume decreasing to zero. Then, we conclude that:

$$\lim_{i\to\infty} \frac{V(E_i,G)}{d_m(E_i,G)^n} = \frac{0}{D(G)^n} = 0.$$

□

Theorem 2 *Let G be an open convex body and E a subset of G obtained by a hyperplane cut. Then,*

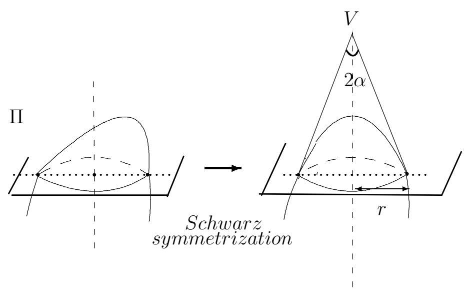
$$\frac{V(E,G)}{d_m(E,G)^n} \geq 0. \tag{1}$$

PROOF. We can distinguish two cases:
Case 1: G is strictly convex:

As G is a strictly convex body, we can choose a hyperplane Π so that if E is a subset of G obtained by the intersection with the half-space determined by Π, in all the points of $\partial E \setminus (G \cap \partial E)$ there exists a tangent hyperplane.

Let us consider a straight line t orthogonal to the hyperplane Π; we can apply the Schwarz symmetrization with respect to t to the subset E and we obtain a new set E' of revolution which has the same volume than E and such that the image of the relative boundary of E under Schwarz symmetrization is a (n-1)-dimensional ball with radius r. Moreover, this symmetrization does not increase the diameter, so $D(E) \geq D(E')$.

The body of revolution E' is contained in a n-dimensional cone C with vertex V, which obviously has greater volume than E'.



Finally, we consider a sequence of parallel hyperplanes to Π so that $V(E')$ decreases to zero and the half angle at the vertex V goes to $\pi/2$. Then,

$$\lim_{\alpha\to\pi/2} \frac{V(E,G)}{d_m(E,G)^n} \leq \lim_{\alpha\to\pi/2} \frac{V(E')}{(2r)^n} \leq$$

$$\lim_{\alpha\to\frac{\pi}{2}} \frac{\pi^{\frac{(n-1)}{2}} \cot\text{an } \alpha}{n 2^n \Gamma(\frac{n-1}{2}+1)} = 0.$$

Consequently, the inequality (1) holds.

Case 2: G is not strictly convex:

If G is not strictly convex there exists a straight line segment t in the boundary of G. We consider a sequence of hyperplanes Π_i parallel to t so that the volume of the subsets E_i determined by the intersections with Π_i decreases to zero. Then,

$$\lim_{i\to\infty} \frac{V(E_i,G)}{d_m(E_i,G)^n} \leq \frac{0}{(length(t))^n} = 0.$$

$\square$

The following proposition provides an upper bound for the ratio between the relative area and the n-power of the minimum relative diameter.

Theorem 3 *Let G be an open convex body and E a subset of G such that E and $G \setminus E$ are connected. Then,*

$$\frac{V(E,G)}{d_m(E,G)^n} \leq \frac{\pi^{n/2}}{\Gamma(1+\frac{n}{2})2^n}$$

PROOF. Let $B(r)^n$ be a ball with radius r such that:

$$V(B(r)^n) \leq \frac{V(G)}{2} \Longleftrightarrow r \leq \left(\frac{V(G)}{2\omega_n}\right)^{1/n}$$

then, as a consequence of the isodiametric inequality (see [1]):

$$\frac{V(E,G)}{d_m(E,G)^n} \leq \frac{\pi^{n/2}}{\Gamma(1+\frac{n}{2})2^n}.$$

$\square$

3. Relative isodiametric inequalities concerning the relative area and the maximum relative diameter of a subset of a convex body

The aim of this section is to maximize and minimize the ratio between the relative area and the n-power of the maximum relative diameter of a subset E of G.

Theorem 4 *Let G be an open bounded convex set in $\mathbb{R}^n$ and E a subset of G such that E and $G \setminus E$ are connected. Then,*

$$\frac{V(E,G)}{d_M(E,G)^n} \geq 0.$$

PROOF. Let G be an open convex body in the Euclidean space. We can suppose without lost of generality that $0 \in G$. Let us consider the sequence $\{E_i\}_{i=2}^{\infty}$ where each $E_i = \frac{1}{i}G$.

If we compute the ratio between the relative volume and the n-power of the maximum relative diameter, the limit of this ratio is 0 when $i \to \infty$. In fact, the relative volume decreases to 0 and $d_M(E_i,G) = D(G \setminus E_i)$ is the diameter of G for all i:

$$\lim_{i\to\infty} \frac{V(E_i,G)}{d_M(E_i,G)^n} = \frac{0}{D(G)^n} = 0.$$

$\square$

Theorem 5 *Let G be an open convex body and E a subset of G obtained by a hyperplane cut. Then,*

$$\frac{V(E,G)}{d_M(E,G)^n} \geq 0.$$

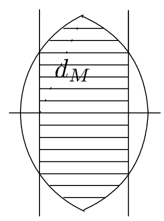
It is easy to prove this theorem using the same argument that in the proof of theorem 2 (when G is not strictly convex), considering $t = D(G)$.

The problem of maximizing the ratio between the relative area and the maximum relative diameter is attached to the so called "fencing problems". Such problems consider dividing a region into two parts of equal volume (area) by a fence. We are going to use some results about fencing problems for proving the following theorem (see [5], [6]).

Theorem 6 *Let G be a planar bounded convex set and E a subset of G obtained by a straight line cut, then:*

$$\frac{A(E,G)}{d_M(E,G)^2} \leq 1.2869...$$

The equality is attained for the optimal body described in the following figure (see [5]).



PROOF. Let l be the straight line dividing G into two regions E and $G \setminus E$, and suppose that $A(E) \leq A(G \setminus E)$. Let us consider two different cases:

1) $d_M(E,G) = D(G \setminus E)$ and 2) $d_M(E,G) = D(E)$.

1) If $d_M(E,G) = D(G\setminus E)$ and $A(E) < A(G\setminus E)$, we translate l till another line l' determining a new division of G into two other regions E' and $G\setminus E'$ in such a way that one of the two following situations hold:

1.1) $A(E') = A(G\setminus E')$ and $d_M(E',G) = D(G\setminus E')$.

Then $A(E',G) = A(E') > A(E) = A(E,G)$ and $d_M(E',G) < d_M(E,G)$ and E' determines a fencing problem. Hence,

$$\frac{A(E,G)}{d_M(E,G)^2} \leq \frac{A(E')}{d_M(E',G)^2} \leq 1.2869...,$$

(For the last inequality see [5] and [6])

1.2) $A(E') < A(G\setminus E')$ and $d_M(E',G) = D(E') = D(G\setminus E')$.

In this case we have $A(E,G) = A(E) \leq A(E') = A(E',G)$ and $d_M(E,G) \geq d_M(E',G)$, so:

$$\frac{A(E,G)}{d_M(E,G)^2} \leq \frac{A(E')}{D(E')^2}, \tag{2}$$

and also,

$$\frac{A(E,G)}{d_M(E,G)^2} \leq \frac{A(G\setminus E')}{D(G\setminus E')^2}. \tag{3}$$

Now we consider the intersection points P and Q of l' with ∂G.

Let E'' be either E' or $G\setminus E'$ where the supporting lines at P and Q make internal angles whose sum is smaller or equal than π. Let us consider the symmetric set of E'' with respect to the middle point O of the segment PQ. Let this set be E'''. $E''\cup E'''$ is a centrally symmetric convex set where the area is $2A(E'')$. It is easy to see that:

$$D(E''') = D(E'') = d_M(E',G) =$$
$$= D(E') = D(G\setminus E').$$

Then, from inequalities (2) and (3),

$$\frac{A(E,G)}{d_M(E,G)^2} \leq \frac{A(E'')}{D(E'')^2} \leq 1.2869...,$$

(For the last inequality see [5]).

2) Suppose that $d_M(E,G) = D(E)$. Then,

$$\frac{A(E,G)}{d_M(E,G)^2} = \frac{A(E)}{D(E)^2},$$

and also,

$$\frac{A(E,G)}{d_M(E,G)^2} \leq \frac{A(G\setminus E)}{D(G\setminus E)^2}.$$

Let E' be either E or $G\setminus E$, where the supporting lines at P and Q realize internal angles whose sum is smaller or equal than π. By a similar argument to that used in the case 1.2 we conclude that

$$\frac{A(E,G)}{d_M(E,G)^2} \leq \frac{A(E'')}{D(E'')^2} \leq 1.2869...$$

□

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