Pointed Encompassing Trees

(Extended Abstract)

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Abstract

It is shown that for any set of disjoint line segments in the plane there exists a *pointed binary encompassing tree*, that is, a spanning tree on the segment endpoints that contains all input segments, has maximal degree three, and such that every vertex is incident to an angle greater than π. As a consequence, it follows that every set of disjoint line segments has a bounded degree pseudo-triangulation.

1. Introduction

Disjoint line segments in the plane are the fundamentals of computational geometry. They form the atomic structure of most planar geometric data structures and geographic information systems. Planar objects are typically represented by a polygonal approximation which, in turn, is composed of (interior) disjoint line segments. Not surprisingly, researchers studied many of their combinatorial properties, such as visibility, compact representation, and ray shooting.

Geometric graphs. We follow one particularly well-studied trail: that of constrained geometric graphs. A *geometric graph* is a graph together with a planar embedding such that the edges are straight line segments. We consider *crossing-free* geometric graphs, that is, we do not allow two edges to cross. Observe that crossing-free is not equivalent to *planar*, since planar graphs may have embeddings in the plane with crossing edges. Given a set of disjoint segments in the plane (that is, a crossing-free geometric matching), we say that a graph is *encompassing* if it is a connected crossing-free geometric graph that contains all input segments as edges (without Steiner points).

It is known that there does not always exists a Hamiltonian circuit (nor path) through a set of disjoint segments. In fact, it is NP-complete to decide if a Hamiltonian circuit exists for a given set of segments, if the segments are allowed to intersect at their endpoints [16]. Rappaport et al. [17] gave a polynomial time algorithm for a set of convexly independent segments. Among n disjoint segments in the plane there are always $\Theta(\log n)$ for which an encompassing path exists [7], this number amounts to $\Theta(\sqrt{n})$ if all segments are axis-parallel [21].

The maximal degree of an encompassing tree on the segment endpoints that is *constrained* to contain all input segments is, therefore, at least three. After a preliminary upper bound of seven by Bose and Toussaint [6], Bose et al. [5] proved that an encompassing tree with maximal degree three always exists. Later Hoffmann and Tóth [8] showed that there is also a *Hamiltonian* encompassing graph with maximum degree three.

Pointedness. Pseudo-triangulations are decompositions of the plane invented by Pocchiola and Vegter [14]. A pseudo-triangulation is a partition of the convex hull of input points or polygonal objects into pseudo-triangles, that is, simple polygons with exactly three vertices whose interior angle is less than or equal to π. They obtained considerable attention recently, as they have found numerous important applications in visibility [13,14], rigidity [20], kinetic collision detection [1,11], and guarding [19]. Pseudo-triangulations, just like triangulations, are also crossing-free geometric graphs. A characteristic property of *minimal* pseudo-triangulations is that for every vertex p all the incident edges are on one side of a line through p (in other words, every vertex is incident to an angle greater than π). Using Streinu's terminology [20], this property is called *pointedness.*

By a result of Streinu [20], there is an encompassing pointed pseudo-triangulation for any set of disjoint line segments: We obtain a pseudo-triangulation by adding edges greedily while pointedness is maintained. Rote et al. [18] recently studied the size of minimum pseudo-triangulations *constrained* to contain a set of *non-crossing* segments. The pointed (or equivalently, minimum) pseudo-triangulation of n *disjoint* segments always has $4n - 3$ edges and $2n - 2$ faces.

2. Results

On one hand, both the algorithm of Hoffmann and Tóth [8] and that of Bose et al. [5] are doomed to violate pointedness due to their proof techniques. In fact, the algorithm in [8] can output a binary encompassing graph for which no spanning subgraph is pointed. On the other hand, the pointed encompassing tree obtained by the straightforward method of Streinu [20] has no guarantee on the degree of the resulting geometric graph. Here, we show how to construct an encompassing tree that respects pointedness *and* has maximal vertex degree at most three:

Theorem 1 *For any set of disjoint line segments in the plane there exists a pointed binary encompassing tree.*

An application. It is known that the triangulation of a planar point set can have arbitrarily high degree. This is also true for triangulations constrained to contain disjoint line segments. Kettner et al. [10] proved that for any set of points in the plane there is a pseudo-triangulation with maximal degree at most five. Bounded vertex degree is a useful property in most applications, as local operations or kinetic data structures require a constant amount of updates. Recently, Aichholzer et al. [2] showed that a bounded degree pseudo-triangulation constrained to contain a Hamiltonian circuit (a simple polygon) also exists, with a degree bound of ten. We can extend these results to pseudo-triangulations constrained to contain disjoint line segments (that is, a perfect matching).

Theorem 2 *Every set of disjoint line segments has a pointed pseudo-triangulation with maximum vertex degree at most ten.*

The best lower bound we could generate is a set of disjoint segments such that in any pointed pseudo-triangulation there is a vertex of degree at least six.

3. Proof technique

We define a class of weakly simple polygons that we call *pearl polygons.* Every vertex of a pearl polygon has either degree two or degree four. Moreover, for every degree four vertex we mark one incident edge such that deleting all marked edges from the pearl polygon results in a spanning tree. The convex hull of the segments belongs to the class of pearl polygons. We start out from the convex hull, and modify it locally using geodesic curves while maintaining a pearl polygon until certain conditions are satisfied. The proof is completed by applying induction in each face of a suitable convex subdivision of the interior of the pearl polygon.

The general scheme of the induction and the use of geodesic curves are similar to the proof techniques applied in [8]. However, the details are quite different and there are too many to list them within the scope of this abstract. Hence, we have to refer the reader to the full paper at this point.

4. Bounded degree pseudo-triangulations for disjoint segments

A more careful analysis reveals that the following form of Theorem 1 also holds:

Theorem 3 *For any set S of disjoint line segments in the plane there exists a pointed binary encompassing tree such that the maximal degree is at most three, and if a convex hull vertex has degree three then at least one of the incident edges is part of the convex hull.*

We combine this theorem with an algorithm of Aichholzer et al. [2], according to which a simple polygon can be pseudo-triangulated such that the degree of every convex vertex is at most four and the degree of every reflex vertex is at most five, that is, every convex (reflex) vertex has at most two (three) new incident edges in addition to the two incident polygon edges. This result also holds for weakly simple polygons that may have reflex interior angles of 2π.

Let us consider the union of the binary pointed encompassing tree T claimed by Theorem 3 and the convex hull conv(S) of the segments in S. The tree T partitions the polygonal domain conv(S) into *weakly* simple polygons. We obtain a bounded degree pseudo-triangulation by pseudo-triangulation each polygon using the algorithm of Aichholzer et al. [2]. Every interior vertex p_{int} has degree three, and one of the incident angular domains is greater than π. So the degree of p_{int} increases by at most $3 + 2 + 2$ to at most 10. Every convex hull vertex p_{hull} has degree at most 4 in $T \cup \text{conv}(S)$ according to Theorem 3, and the one reflex angular domains lies at the exterior of conv(S) which does not need to be pseudo-triangulated. Therefore the degree of p_{hull} increases by at most $2 + 2 + 2$ to at most 10. This proves Theorem 2.

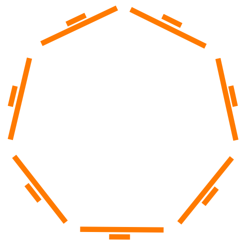

Fig. 1. Lower bound construction.

In the remainder, we describe a set of 14 disjoint segments whose every pseudo-triangulation has a vertex of degree at least 6. The segment endpoints form a regular 28-gon $P = (p_1, p_2, \ldots, p_{28})$. Place seven disjoint segments along the sides $p_{4k+2}p_{4k+3}$, and along parallel the diagonals $p_{4k+1}p_{4k+4}$ for $k = 1, 2, \ldots, 7$, see Fig. 1. The pseudo-triangulation of a convex polygon is also a triangulation. Any triangulation of the inner 14-gon either has a vertex of degree five, or it has three consecutive vertices, each of degree four. Taking the seven outer segments into account adds an additional convex hull edge to each vertex. Moreover, out of any three consecutive vertices of the inner 14-gon one gets another edge, since the outer quadrilaterals have to be (pseudo-)triangulated by any diagonal. Therefore, there is a vertex of degree at least six in P.

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