INDAM Workshop on Holomorphic Iteration, Semigroups, and Loewner Chains

Rome, 9-12 September 2008

LINEARIZATION MODELS FOR PARABOLIC TYPE DYNAMICAL SYSTEMS VIA ABEL'S FUNCTIONAL EQUATIONS

DAVID SHOIKHET

The Galilee Research Center for Applied Mathematics of ORT Braude College, Karmiel

1 Introduction

Let Δ be the open unit disk in the complex plane $\mathbb{C}$ and let Ω be a subset of $\mathbb{C}$. By $Hol(\Delta, \Omega)$ we denote the set of all holomorphic functions (mappings) from Δ into Ω.

Linearization models for continuous semigroup of holomorphic mappings in varions settings have an extensive history, begining with the study of continuous stochastic Markov's branching processes (see, for example, T.E. Harris' book, 1963), Kolmogorov's backward equaition in probability theory and functional-differential equations.

The linearization models for one-parameter semigroups have also proved to be very useful in the study of composition operators and their spectra (see, for example, E. Berkson and H. Porta, 1978, C.C. Cowen, 1981,C.C. Cowen and B.D. MacCluver, 1995, A. Siskakis, 1996).

In the years since classic works much has been written.

A deep investigation of the behaviour of one-parameter semi-groups near their boundary Denjoy-Wolff point based on geometric properties of the model obtained by Abel's functional equction was recently done by M.Contreras and S. Diaz-Madrigal (2005).

It turns out that in those settings solutions of Abel's functional equations are univalent functions convex in one-direction.

In a parallel way, the class of functions convex in one direction has been studied by many mathematicians (for example, W. Hengartner and G. Schober,1970; K. Ciozda, 1980; K. Ciozda, 1979; A. W. Goodman, 1983 and A. Lecko, 2002) as a subclass of the class of functions introduced by Robertson in 1936.

Following this point of view we study some additional geometric and analytic properties of functions convex in one direction by using the asymptotic behavior of one parameter semigroups near their attractive (or repelling) boundary fixed points.

In an opposite way this gives us some new information on some special (but very wide and important classes of complex dynamical systems generated by holomorphic functions having fractional derivatives at their Denjoy-Wolff points.

Definition 1 . *A univalent function* $h \in Hol(\Delta, \mathbb{C})$ *is said to be convex in the positive direction of the real axis if for each* $z \in \Delta$ *and* $t \geq 0$, *the point* $h(z) + t$ *belongs to* $h(\Delta)$.

It is well known that for each $z \in \Delta$, the limit

$$\lim_{t \to \infty} h^{-1}(h(z) + t) =: \zeta \qquad (1)$$

exists and belongs to $\partial\Delta$.

Moreover, since the family $S = \{F_t\}_{t \geq 0}$ defined by

$$F_t(z) = h^{-1}(h(z) + t) \tag{2}$$

forms a one-parameter continuous semigroup of holomorphic self-mappings of Δ, it follows from the continuous version of the Denjoy-Wolff Theorem (see E. Berkson and H. Porta, 1978; S.Reich and D. Shoikhet, 1997 and D. Shoikhet, 2001) that the limit point ζ in (1) is unique and does not depend on $z \in \Delta$.

Without loss of generality we may set $\zeta = 1$.

We denote by $\Sigma[1]$ the class of functions convex in the positive direction of the real axis, normalized by the conditions

$$\lim_{t\to\infty} h^{-1}(h(z)+t) = 1 \quad \text{and} \quad h(0) = 0. \tag{3}$$

In the reverse direction one can assign to each semigroup $S = \{F_t\}_{t\geq 0}$ of holomorphic self-mappings of Δ with a boundary Denjoy-Wolff point $\zeta = 1$, a univalent function $h \in Hol(\Delta, \mathbb{C})$ which is a common solution of Abel's functional equation

$$h(F_t(z)) = h(z) + t, \quad z \in \Delta, t \geq 0, \tag{4}$$

and hence is convex in the positive direction of the real axis.

The set $h(\Delta)$ is called a planar domain for S and the pair $(h, h(\Delta))$ is said to be a linearization model for S.

To be more precise, we recall that by the Berkson-Porta theorem (1978) (see also M. Abate, 1992; S. Reich and D. Shoikhet, 1996; 1997 and 2005),

for each one-parameter continuous semigroup $S = \{F_t\}_{t\geq 0}$ of holomorphic self-mappings of Δ, the limit

$$f(z) := \lim_{t\to 0^+} \frac{z - F_t(z)}{t} \tag{5}$$

exists.

This limit function is called the infinitesimal generator of S.

Moreover, the semigroup $S = \{F_t\}_{t\geq 0}$ can be reproduced as the solution of the Cauchy problem

$$\begin{cases} \dfrac{\partial u(t,z)}{\partial t} + f(u(t,z)) = 0 \\ u(0,z) = z, \end{cases} \tag{6}$$

where we set $F_t(z) := u(t,z)$, $t \geq 0$, $z \in \Delta$.

By $G[1]$ we denote the class of functions which consists of all holomorphic generators f of continuous semigroups having the Denjoy-Wolff point $\zeta = 1$. In this case f admits the Berkson-Porta representation

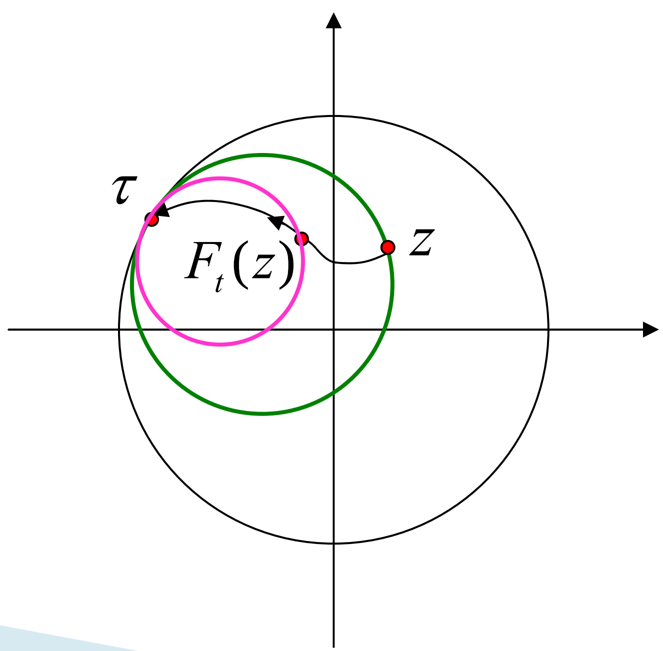
$$f(z) = -(1-z)^2 p(z), \quad z \in \Delta,$$

with $\operatorname{Re} p(z) \geq 0$ everywhere.

In addition, if $S = \{F_t\}_{t\geq 0}$ is the semigroup generated by f (S is defined via the Cauchy problem (9), where we set $F_t(z) := u(t,z)$, $t \geq 0$, $z \in \Delta$), then $\lim_{t\to\infty} F_t(z) = 1$ and for each $t \geq 0$,

$$\angle \lim_{z\to 1} \frac{\partial F_t(z)}{\partial z} = e^{-t\beta},$$

$$\frac{|F_t(z)-\tau|^2}{1-|F_t(z)|^2} \leq \exp\left(-t_\beta\right)\frac{|z-\tau|^2}{1-|z|^2}$$



where $\beta := \angle\lim_{z\to 1}\dfrac{f(z)}{z-1} = \angle\lim_{z\to 1} f'(z) \geq 0$ (see, for example, M. Elin and D. Shoikhet, 2001 and M. D. Contreras, S. Díaz-Madrigal and Ch. Pommerenke, 2006).

If $\beta > 0$, then S and its generator f are said to be of hyperbolic type. Otherwise, that is, if $\beta = 0$, the semigroup S and its generator f are said to be of parabolic type.

Conversely, it is shown in M. Elin and D. Shoikhet (2001), that if f is a holomorphic generator on Δ such that the angular limit

$$\angle \lim_{z \to 1} \frac{f(z)}{z-1} =: f'(1) \qquad (7$$

exists finitely with $\operatorname{Re} f'(1) \geq 0$, then $f'(1)$ is, in fact, a real number

f has no null points in Δ and belongs to $G[1]$.

This, in turn, implies that the formula

$$h(z) = -\int_0^z \frac{dz}{f(z)}$$

establishes a one-to-one correspondence between the classes $G[1]$ and $\Sigma[1]$.

In this work we first are interested in finding simple analytic conditions which determine certain geometric properties of functions h in the class $\Sigma[1]$, such as the location of the image $h(\Delta)$ in either a half-plane or a strip, and its containing either a half plane or a strip.

In the context of semigroup theory these geometric questions may be interpreted as follows:

Is a given one-parameter continuous semigroup an outer or inner,

conjugates of a group of automorphisms (see the definition below). In other words, the problem is finding a fractional linear model of the semigroup which is defined by a group of automorphisms of Δ.

To be more precise, we need the following definition.

Definition 2 . *Let $S = \{F_t\}_{t\geq 0}$ be a one-parameter continuous semi-group of holomorphic self-mappings of Δ.*

- *A univalent mapping $\psi : \Delta \to \Delta$ is called an outer conjugator of S if there is a one-parameter semigroup $\{G_t\}_{t\geq 0} \subset Hol(\Delta, \Delta)$ of linear fractional transformations of Δ such that*

$$\psi \circ F_t = G_t \circ \psi. \tag{8}$$

- *A univalent mapping $\varphi : \Delta \to \Delta$ is called an inner conjugator of S if there is a one-parameter continuous semigroup $\{G_t\}_{t\geq 0} \subset Hol(\Delta, \Delta)$ of linear fractional transformations of Δ such that*

$$F_t \circ \varphi = \varphi \circ G_t. \tag{9}$$

Note that by the Linear Fractional Model theorem (see P. S. Bourdon and J. H. Shapiro, Cyclic phenomena for composition operators, Mem. Amer. Math. Soc. 125 (1997)) for each holomorphic self-mapping there exists a conjugating function $G : \Delta \to \mathbb{C}$ which is not necesserely a self-mapping of Δ.

However, in our considerations the main point in Definition 2 is that the conjugators φ and ψ are self-mappings of Δ.

It is rather transparent fact that if $S = \{F_t\}_{t\geq 0}$ is a semigroup with a boundary Denjoy-Wolff point $\zeta \in \partial\Delta$ (say $\zeta = 1$) and h is the solution of Abel's equation (4) normalized by (9), then an outer (respectively, inner) conjugator of S exists if and only if $h(\Delta)$ lies in (respectively, contains) a half-plane Π.

In both cases the conjugate semigroup $\{G_t\}_{t\geq 0}$ of linear fractional transformations (see formulas (8) and (9), is a group of automorphisms of Δ if and only if Π is horizontal, i.e., its boundary is parallel to the real axis.

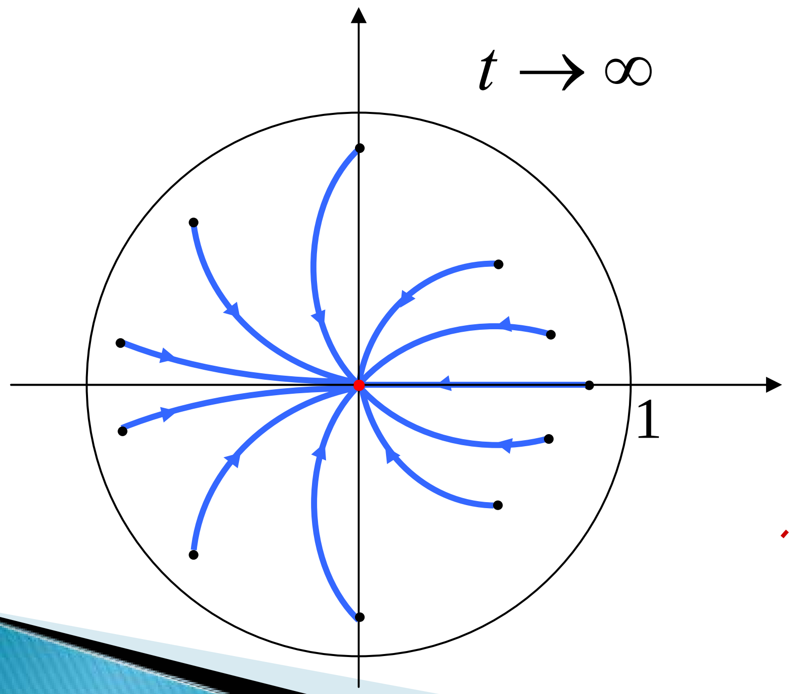
The latter situation is of special interest for inner conjugation since equation (9) means that the semigroup $S = \{F_t\}_{t\geq 0}$ forms a group of automorphisms of the domain $\varphi(\Delta)$ which is called a backward flow invariant domain for S (see M. Elin, D. Shoikhet and L. Zalcman, 2008). We will concentrate on this issue later for semigroup generators which are fractionally differentiable at their boundary Denjoy-Wolff points. In the mean time we need the following observation.

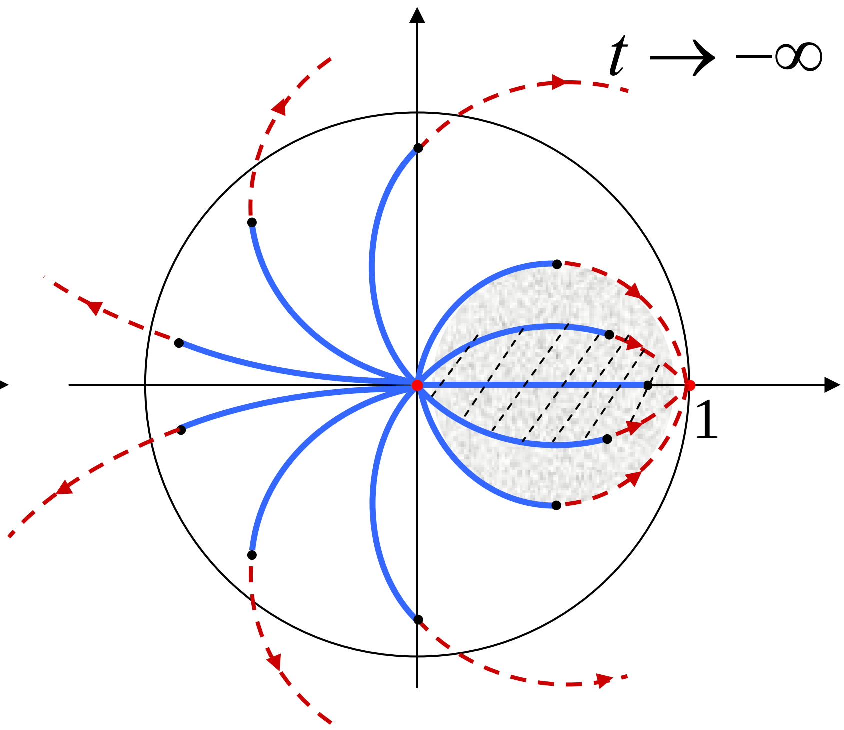
Examples: 1. Shrinking the disk to a smaller disk:

$$F_t(z) = \frac{e^{-t} z}{(e^{-t} - 1)z + 1}$$

$\tau_1 = 0, \ \tau_2 = 1$ - the common fixed points

$\tau_1 = 0$ - the DW point



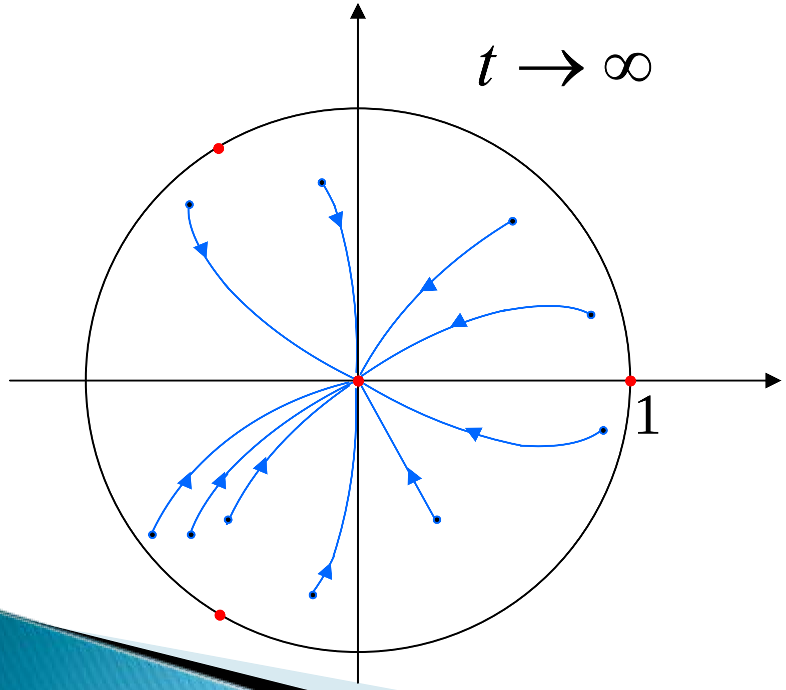


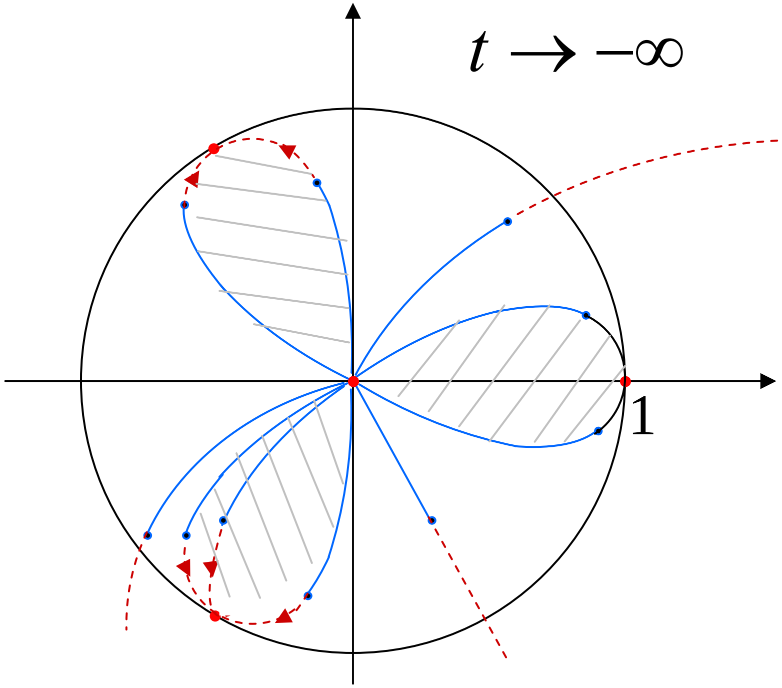
2. $$F_t(z) = \frac{ze^{-t}}{\sqrt[n]{1 - z^n + z^n e^{-nt}}}$$

$n = 3$

$\tau_0 = 0, \;\; \tau_k = e^{i\frac{2\pi k}{3}}, \;\; k = 1,2,3$ - the common fixed points

$\tau_0 = 0$ - the DW point



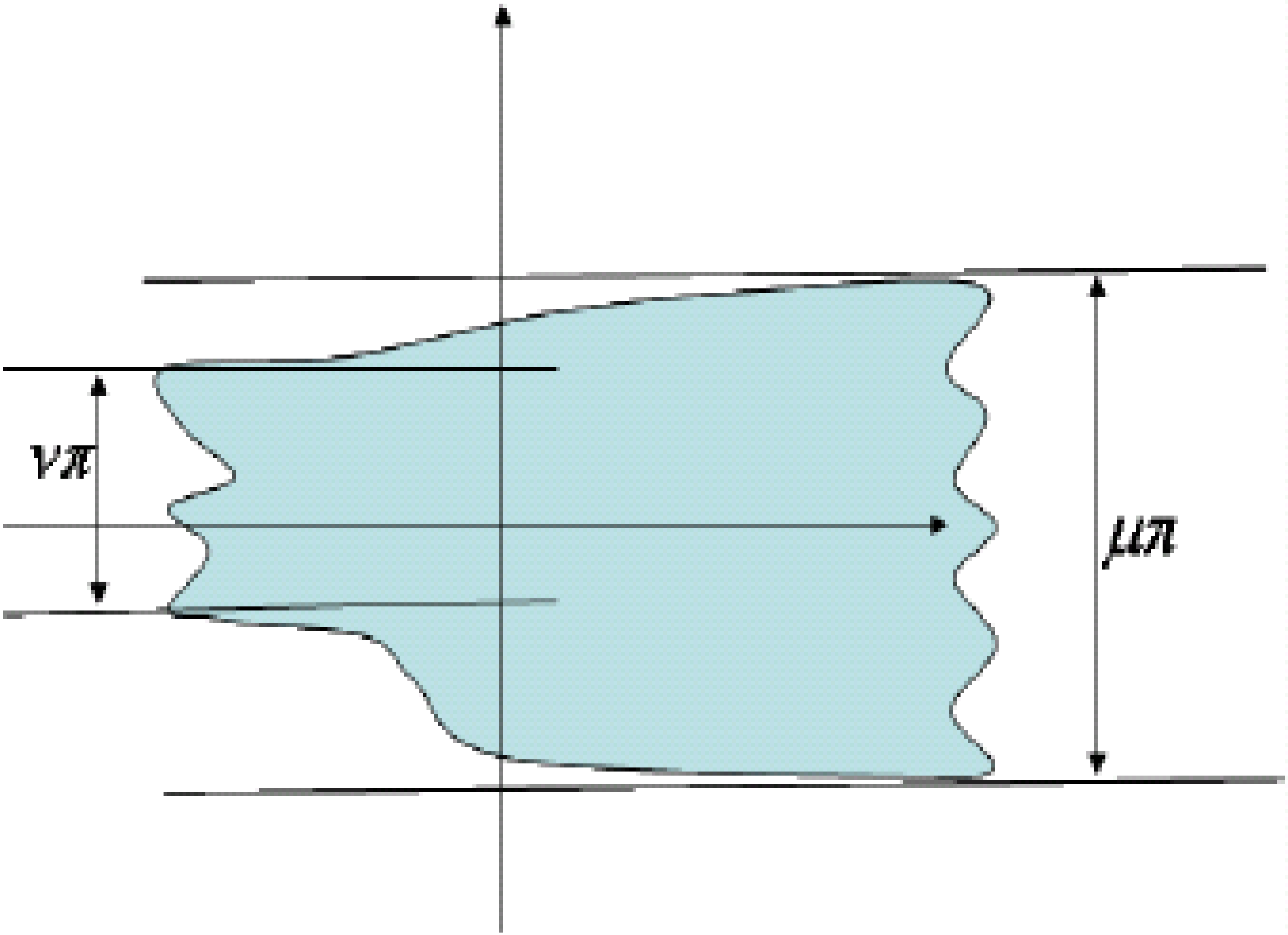


It can be shown that if h belongs to $\Sigma[1]$, then the angular limit

$$\mu := \angle \lim_{z \to 1} (1 - z) h'(z) \tag{10}$$

exists and belongs to $(0, \infty]$.

It turns out that this limit μ is finite if and only if $|\operatorname{Im} h(z)|$ is bounded. Moreover, the number $\pi\mu$ is the width of the minimal horizontal strip which contains $h(\Delta)$.



$\nu\pi$
$\mu\pi$

If μ in (10) is infinite, it may happen that for some $\alpha > 0$, the angular limit

$$\angle \lim_{z \to 1} (1-z)^{1+\alpha} h'(z) =: \mu, \tag{11}$$

or even the unrestricted limit

$$\lim_{\substack{z \to 1 \\ z \in \Delta}} (1-z)^{1+\alpha} h'(z) =: \mu, \tag{12}$$

exist finitely.

The latter condition, for example, follows from the geometric property of $h(\Delta)$ to have a Dini smooth corner of opening $\pi\alpha$ at infinity for some $\alpha \in (0, 2]$ (see Ch. Pommerenke, Boundary Behavior of Conformal Maps).

Therefore it is natural to consider the subclasses $\Sigma_A^\alpha[1]$, respectively, $\Sigma^\alpha[1]$ $(\Sigma^\alpha[1] \subset \Sigma_A^\alpha[1])$, $\alpha \geq 0$, which consist of those $h \in \Sigma[1]$ satisfying (11), respectively, (12).

2 Subordination theorems (outer conjugation).

2.1 Auxiliaries results.

Proposition 3 . *A function* $h \in Hol(\Delta, \mathbb{C})$, $h \neq 0$, *with* $h(0) = 0$, *belongs to* $\Sigma[1]$ *if and only if*

$$\operatorname{Re}\left[(1-z)^2 h'(z)\right] \geq 0 \tag{13}$$

and if and only if $f \in Hol(\Delta, \Omega)$, *defined by* $f(z) = -\dfrac{1}{h'(z)}$, *belongs to* $G[1]$.

Proposition 4 . *Let* $\{F_t\}_{t\geq 0}$ *be the semigroup with the Denjoy-Wolff point* $z=1$, *such that* $F_t(z) = h^{-1}(h(z+t))$, $h \in \Sigma[1]$. *Then* $h(\Delta)$ *lies in a horizontal half-plane if and only if there is a group* $\{G_t\}$, $t \in \mathbb{R}$ *of parabolic automorphisms of* Δ *and a conformal self-mapping* ψ *of* Δ *such that for all* $t \geq 0$ *the semigroup* $\{F_t\}$ *outer conjugates with* $\{G_t\}$:

$$\psi(F_t(z)) = G_t(\psi(z)), \quad t \geq 0.$$

Remark 5 . *In particular, each hyperbolic group can conjugate with a parabolic one.*

To proceed we need the following lemma.

Lemma 6 . *Let* $h \in \Sigma[1]$. *Then the angular limit*

$$\angle \lim_{z \to 1} (1 - z) h'(z) := \mu$$

exists and is either a positive real number or infinity.

2.2 Main results.

First we turn to a description of the class $\Sigma_A^0[1]$.

Theorem 7 . *Let $h \in \Sigma[1]$. The following assertions are equivalent:*

(i) $h \in \Sigma_A^0[1]$, *i.e.,*

$$\angle \lim_{z \to 1} (1-z)h'(z) = \mu$$

exists finitely;

(ii) $\Omega = h(\Delta)$ *lies in a horizontal strip;*

(iii) h *is a Bloch function, i.e.,*

$$\|h\|_{\mathcal{B}} := \sup_{z \in \Delta} (1-|z|^2)\,|h'(z)| < \infty.$$

Moreover, in this case,

(a) the smallest strip which contains $h(\Delta)$ *is*

$$\left\{w \in \mathbb{C} : |\operatorname{Im} w - a| < \frac{\pi\mu}{2}\right\},$$

where

$$a = \lim_{r \to 1^-} \operatorname{Im} h(r).$$

(b) $2\,|\mu| \leq \|h\|_B \leq 4\,|\mu|.$

Remark 8 . *Once the equivalence of (i)-(iii) and assertion (a) are proved, we have by the Köbe Distortion Theorem the following estimate:*

$$\frac{\pi\,|\mu|}{2} \leq \|h\|_{B} \leq 2\pi\,|\mu|\,. \qquad (14)$$

Clearly, assertion (b) improves (14).

Such estimates are very important in finding non-trivial bounds for integral means of $|h(z)|$, as well as its even powers (see, for example, Ch. Pommerenke, Boundary Behavior of Conformal Maps).

Now we start to study the classes $\Sigma_A^\alpha[1]$, respectively, $\Sigma^\alpha[1]$, $\alpha > 0$, of functions $h \in \Sigma[1]$, for which

$$\angle \lim_{z \to 1} (1-z)^{1+\alpha} h'(z) =: \mu,$$

exists finitely.

Remark 9 . *Note that these classes arize naturally, if $h(\Delta)$ is contained in a half-plane $\Pi = \{w \in \mathbb{C} : w = h_1(z) := \frac{ibz}{1-z}, \ z \in \Delta\}$ and $\psi = h_1^{-1} \circ h$ has a Dini's smooth corner of opening $\pi\alpha$, $0 < \alpha \leq 1$.*

We will see below, that if $h(\Delta)$ contains a half-plane $h_1(\Delta)$ and $\varphi = h^{-1} \cdot h_1$ has a Dini's smooth corner of opening $\pi\gamma$, then $\frac{1}{2} \leq \gamma \leq 1$ and $h \in \Sigma_A^\alpha[1]$ with $\alpha = \frac{1}{\gamma} \in [1, 2]$.

- We say that a generator f belongs to the class $G_A^\alpha[1]$ $(G^\alpha[1])$, $\alpha > 0$, if

$$\angle \lim_{z \to 1} \frac{f(z)}{(1-z)^{1+\alpha}} = a \neq 0 \tag{15}$$

$$\left(\lim_{\substack{z \to 1 \\ z \in \Delta}} \frac{f(z)}{(1-z)^{1+\alpha}} = a \neq 0 \right). \tag{16}$$

It is clear that $h \in \Sigma_A^\alpha[1]$ (respectively, $\Sigma_A^\alpha[1]$) if and only if f defined by $f(z) = -\frac{1}{h'(z)}$, belongs to the class $G_A^\alpha[1]$ (respectively, $G^\alpha[1]$) with $a = -\frac{1}{\mu}$. Of course, if $\alpha > 0$, then f is of parabolic type.

The following example shows that the class $\Sigma^\alpha[1]$ (respectively, $G^\alpha[1]$) is a proper subset of the class $\Sigma_A^\alpha[1]$ (respectively, $G_A^\alpha[1]$).

Example 10 . *Consider the function* $f \in Hol(\Delta, \mathbb{C})$ *defined by*

$$f(z) = -(1-z)^2 \left[1 - \exp\left(-\frac{1+z}{1-z}\right)\right]^{\beta} (1-z)^{1-\beta}$$

$0 < \beta \leq 1$.

It follows by the Berkson-Porta formula, that $f \in G[1]$. *Moreover,* $f \in G_A^{\alpha}[1]$, $\alpha = 2 - \beta$, *since the angular limit*

$$\angle \lim_{z \to 1} \frac{f(z)}{(1-z)^{1+\alpha}} = -1.$$

At the same time the unrestricted limit

$$\lim \frac{f(z)}{(1-z)^{1+\alpha}}$$

does not exist, i.e., $f \notin G^{\alpha}[1]$.

Theorem 11 . *Let* $h \in \Sigma_A^\alpha[1]$ *with*

$$\angle \lim_{z \to 1} (1-z)^{1+\alpha} h'(z) =: \mu.$$

The following assertions hold:

(A) $\alpha \in [0, 2]$.

(B) Assume that for some $z \in \Delta$, *the trajectory* $\{F_t(z)\}_{t \geq 0}$, *where* $F_t(z) = h^{-1}(h(z) + t)$, *converges to* $z = 1$ *nontangentially.*

Then

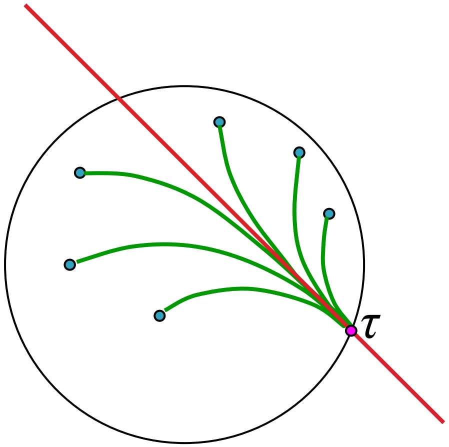
(i) $|\arg \mu| \leq \dfrac{\pi}{2} \min\{\alpha, 2-\alpha\}$. *Moreover, if* $\alpha \in (0, 1]$, *then this inequality is sharp;*

(ii) $\lim_{t\to\infty} t(1 - F_t(z))^\alpha = \dfrac{\mu}{\alpha}$ *for all* $z \in \Delta$;

(iii) $\lim_{t\to\infty} \arg(1 - F_t(z)) = \dfrac{1}{\alpha} \arg \mu$ *for all* $z \in \Delta$.

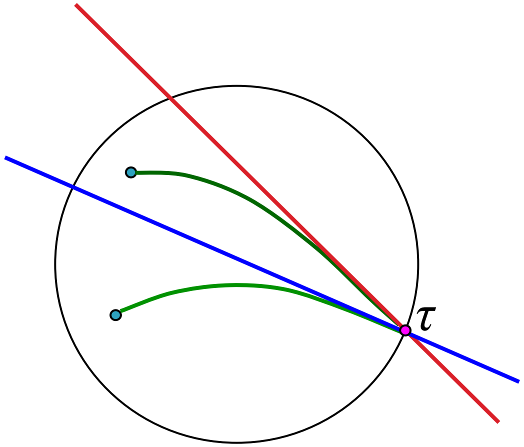
✓ Parabolic case

All parabolic semigroups which have the same fractional angular derivative at their common Denjoy-Wolff point share the same asymptote, irrespective of their initial points.



✓ Hyperbolic case

Each hyperbolic semigroup trajectory has an asymptote at its Denjoy-Wolff point, which depends on the initial point of the trajectory.



.

Remark 12 . *We note that the conclusion of Theorem 11 remains, of course, true if we replace condition (15) by (16). Moreover, the requirement of nontangential convergence in this case is not necessary. In addition, formulas (i) and (iii) show that if* $\alpha > 1$ $(\alpha \leq 2)$*, then all the trajectories must necessarily converge in a nontagential way.*

This proves the following assertion.

Corollary 13 . *Let* $f \in G^{\alpha}[1]$*. The following assertions are equivalent*

(i) for some $z \in \Delta$*, the trajectory* $\{F_t(z)\}_{t \geq 0}$ *converges to* $z = 1$ *tangentially;*

(ii) for all $z \in \Delta$*, the trajectories* $\{F_t(z)\}_{t \geq 0}$ *converge to* $z = 1$ *tangentially;*

(iii) $|\arg(-a)| = \dfrac{\pi}{2}\alpha.$

Moreover, in this case $0 < \alpha \leq 1$.

Remark 14 . *Note that the nontangential convergence of the trajectory* $\{F_t(z)\}_{t\geq 0}$ *to the Denjoy-Wolff point* $z = 1$ *of the semigroup* S *means that*

$$\inf_{t\geq 0} \frac{1 - |F_t(z)|}{|1 - F_t(z)|} > 0.$$

We will see that this condition implies that there is no horizontal half-plane which contains $h(\Delta)$.

In an opposite way, the condition

$$\lim_{t\geq 0} \frac{1 - |F_t(z)|}{|1 - F_t(z)|} = 0$$

implies that the trajectory $\{F_t(z)\}_{t\geq 0}$ *converges tangentially to* $z = 1$.

So, the question is whether the latter condition is sufficient to ensure that $h(\Delta)$ *lies in a horizontal half-plane.*

The following Theorem shows that for $h \in \Sigma^\alpha[1]$ the existence of a horizontal half-plane containing $h(\Delta)$ is, in fact, equivalent to a more strong condition that the ratio $\dfrac{1-|F_t(z)|}{|1-F_t(z)|}$ converges to zero faster that $\dfrac{1}{t}$. Moreover, for $\alpha = 1$, we will see that the limit

$$\lim_{t\to\infty} \frac{t(1-|F_t(z)|)}{|1-F_t(z)|} = L(z)$$

exists and is different from zero if and only if all the trajectories converge to $z = 1$ strongly tangentially, i.e., for each $z \in \Delta$ there is a horocycle internally tangent to $\partial\Delta$ at the point $z = 1$ which does not contain $\{F_t(z)\}_{t\geq s}$ for some $s \geq 0$.

We recall that by $\Sigma^\alpha[1]$, $\alpha \in [0,2]$, we denote the subclass of $\Sigma[1]$ which consists of those functions h for which the unrestricted $\lim_{\substack{z\to 1\\ z\in\Delta}} (1-z)^{1+\alpha} h'(z)$ exists finitely.

Theorem 15 . *Let $h \in \Sigma^\alpha[1]$ with $\alpha \in (0,2]$,*

$$\mu := \lim_{\substack{z\to 1\\ z\in\Delta}} (1-z)^{1+\alpha} h'(z) \neq \infty,$$

and let $h^{-1}(h(z)+t) =: F_t(z)$.

Then $h(\Delta)$ lies in a horizontal half-plane if and only if

$$\sup_{t\geq 0} \frac{t\,(1-|F_t(z)|)}{|1-F_t(z)|} =: M(z) < \infty.$$

Moreover, in this case the following assertions hold:

(a) α *must belong to the half-interval* $(0, 1]$*;*

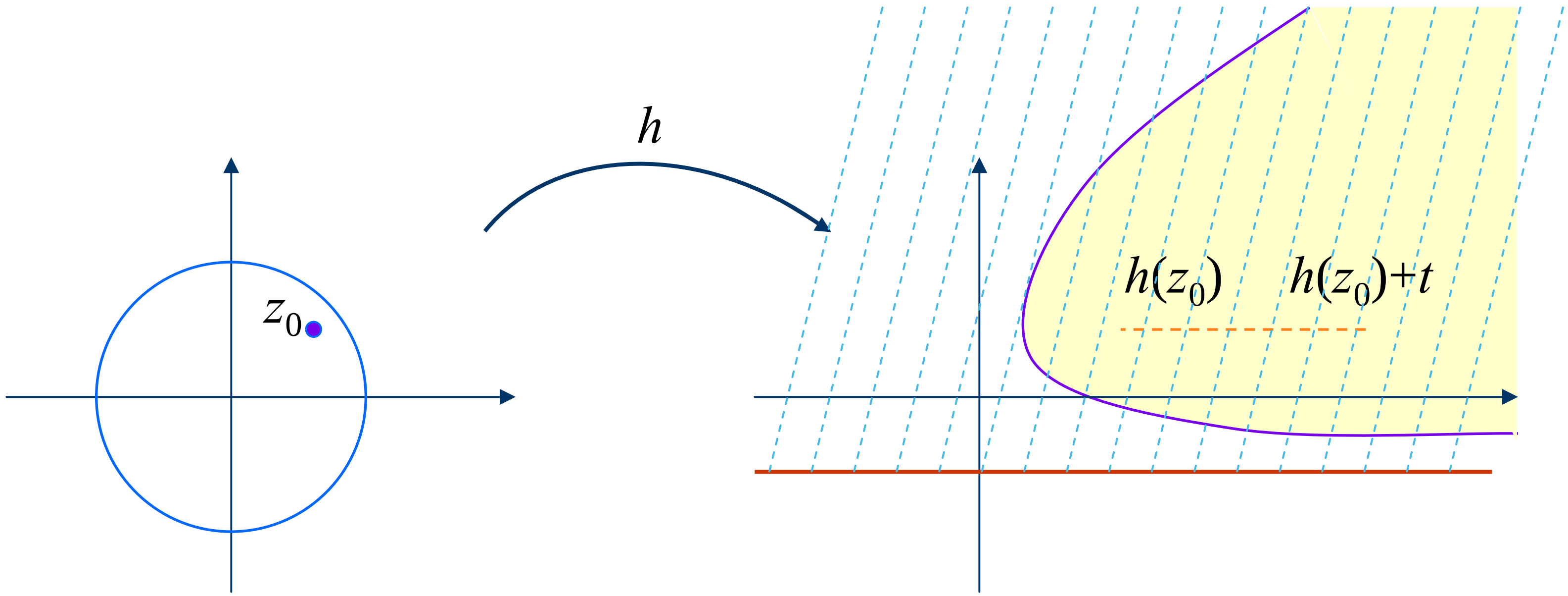
(b) $|\arg \mu| = \frac{\pi}{2}\alpha$*;*

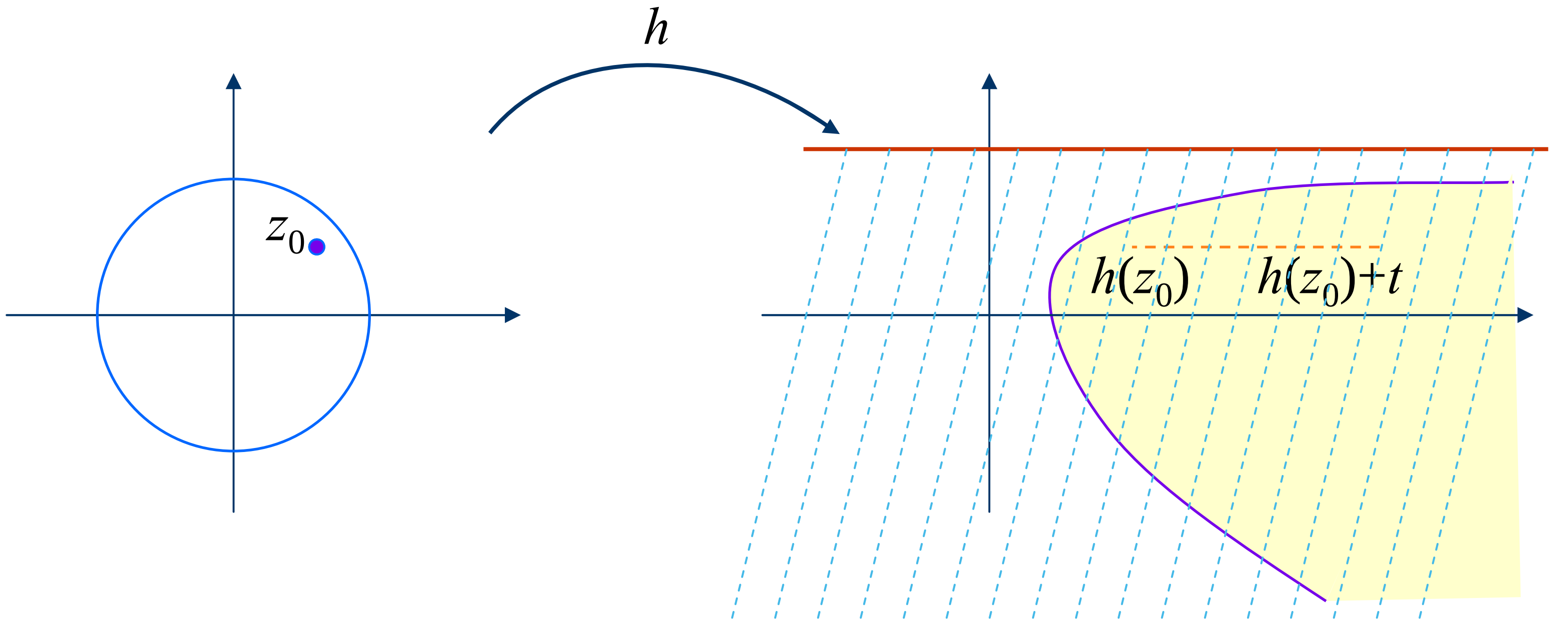
(c) all the trajectories $\{F_t(z)\}$, $t \geq 0$, $z \in \Delta$, *converge tangentially to the point* $z = 1$.

(d) if $\Pi = \{w = \frac{ibz}{1-z}, z \in \Delta\}$ *is a half-plane containing* $h(\Delta)$, *then*

$$b \cdot \arg \mu > 0.$$

In other worlds, $\operatorname{Im} h(z)$ is bounded from above if and only if $\arg \mu > 0$. Other wise $(\arg \mu < 0)$ the set $\operatorname{Im} h(z)$ is bounded from below. Recall that since $\alpha \neq 0$, $\operatorname{Im} h(z)$ cannot be bounded from the both sides.


h
z_0
$h(z_0)$
$h(z_0)+t$



h
z_0
$h(z_0)$
$h(z_0)+t$

Corollary 16 . *Let* $h \in \Sigma^{\alpha}[1]$ *with* $\alpha \in (0,2]$ *and*

$$\mu := \lim_{\substack{z \to 1 \\ z \in \Delta}} (1-z)^{1+\alpha} h'(z) \neq \infty.$$

If either $\alpha > 1$ *or* $|\arg \mu| \neq \frac{\pi}{2}\alpha$, *then there is no horizontal half-plane which contains* $h(\Delta)$.

In particular, if μ *is real, then*

$$\sup_{z \in \Delta} \operatorname{Im} h(z) = \infty$$

and

$$\inf_{z \in \Delta} \operatorname{Im} h(z) = -\infty.$$

Now by using Proposition 4 we get the following assertion.

Theorem 17 . *Let* $S = \{F_t\}_{t\geq 0}$ *be a semigroup of parabolic self-mappings of* Δ *generated by* $f \in G^{\alpha}[1]$, *i.e.,*

$$\lim_{z\to 1} \frac{f(z)}{(1-z)^{1+\alpha}} = a,$$

exists finitely and different from zero.

The following are equivalent:

(i) $\{F_t\}$ *outer conjugates with a group* $\{G_t\}_{t\in\mathbb{R}}$ *of parabolic automorphisms of* Δ, *i.e., there is a conformal self-mapping* $\psi : \Delta \to \Delta$ *such that*

$$\psi(F_t(z)) = G_t(\psi(z)), \quad t \geq 0, \;\; z \in \Delta;$$

(ii)

$$\sup_{t\geq 0} \frac{t(1-|F_t(z)|)}{|1-F_t(z)|} =: M(z) < \infty, \quad z \in \Delta.$$

Moreover, in this case

(a) $\alpha \in (0, 1]$;

(b) $|\arg(-a)| = \frac{\pi}{2}\alpha$;

(c) all the trajectories $\{F_t(z)\}_{t\geq 0}$, $z \in \Delta$, $t \geq 0$, *converge tangentially to the point* $z = 1$.

We denote by $d(z)$ the non-euclidean distance from a point $z \in \Delta$ to the boundary point $\zeta = 1$ defined by

$$d(z) := \frac{|1-z|^2}{1-|z|^2}.$$

The sets $\{d(z) < k\}$, $k > 0$, are horocycles internally tangent to $\partial\Delta$ at $z = 1$. It follows from the Julia-Wolff-Caratèodory theorem for semigroups that for each $z \in \Delta$, the function $d(F_t(z))$ is decreasing, so the limit

$$\lim_{t\to\infty} d(F_t(z)) = \varepsilon(z)$$

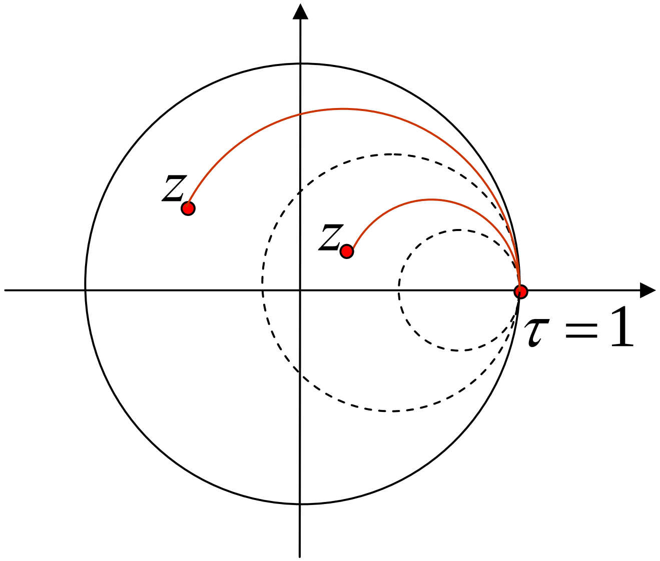
exists and is nonnegative.

It can be shown that either $\varepsilon(z) > 0$ for all $z \in \Delta$ or $\varepsilon(z) = 0$ identically.

Definition 18 . *We say that a semigroup* $S = \{F_t\}_{t\geq 0}$ *is strongly tangentially convergent if* $\varepsilon(z) > 0$.

Geometrically, this means that for each $z \in \Delta$, there is a horocycle E internally tangent to $\partial\Delta$ at $z = 1$ such that the trajectory $\{F_t(z)\}_{t\geq 0}$ lies outside E.

Sometimes a strongly tangentially convergent semigroup is said to be of finite shift (see M. D. Contreras, S. Díaz-Madrigal and Ch. Pommerenke, Second angular derivatives and parabolic iteration in the unit disk, 2005.).


z
z
$\tau = 1$

Theorem 19 . *Let* $h \in \Sigma^{\alpha}[1]$, $\alpha \in [0,2]$ *and let* $S = \{F_t\}_{t \geq 0}$ *be defined by* $F_t(z) = h^{-1}(h(z) + t)$, $z \in \Delta$, $t \geq 0$. *The following assertions hold:*

(A) Assume that for some $z \in \Delta$ *the trajectory* $\{F_t(z)\}_{t \geq 0}$ *converges to* $z = 1$ *strongly tangentially. Then*

(i) $\alpha = 1$;

(ii) $h(\Delta)$ *belongs to a half-plane.*

(B) Conversely. Assume that conditions (i) and (ii) are fulfiled. Then all the trajectories $\{F_t(z)\}_{t \geq 0}$, $z \in \Delta$, *converge to* $z = 1$, *strongly tangentially*

Moreover,

$$\mu = \lim_{z\to 1}(1-z)^2 h'(z)$$

is purely imaginary and either

$$\operatorname{Im} h(z) \geq \frac{-\operatorname{Im}\mu}{2\varepsilon(0)},\ z \in \Delta,\ \textit{if}\ \operatorname{Im}\mu > 0$$

or

$$\operatorname{Im} h(z) \leq \frac{-\operatorname{Im}\mu}{2\varepsilon(0)},\ \textit{if}\ \operatorname{Im}\mu < 0.$$

Proposition 20 . *Let* $S = \{F_t\}_{t\geq 0}$ *be a semigroup of parabolic type generated by* $f \in G^1[1]$, *i.e.*,

$$\lim_{z\to 1} \frac{f(z)}{(1-z)^2} =: \frac{1}{2} f''(1) = a$$

exists finitely and different from zero.

Then $\{F_t\}_{t\geq 0}$ *converges to* $z = 1$ *strongly tangentially if and only if*

$$\lim_{t\to\infty} \frac{t(1-|F_t(z)|)}{|1-F_t(z)|} = L(z)$$

is finite.

Example 21 . *Consider a function $h : \Delta \to \mathbb{C}$ conformally mapping the open unit disk Δ onto the quadrant $\{-1 \le x < \infty, -1 \le y < \infty\}$ given by the formula*

$$h(z) = e^{i\frac{\pi}{4}} \left[\sqrt{\frac{1+z}{1-z}} - 1 \right].$$

It is clear that $h \in \Sigma^{\alpha}[1]$, with $\alpha = \frac{1}{2}$ and

$$\mu = \lim(1-z)^{\alpha+1} h'(z) = e^{i\frac{\pi}{4}}.$$

The corresponding generator $f\ (= -\frac{1}{h'})$ is defined by $f(z) = -(1-z)^2 \sqrt{\frac{1+z}{1-z}} e^{-\frac{\pi}{4}i}$.

Since $h(\Delta)$ lies in a horizontal half-plane all the trajectories $F_t(z)$ must converge to $z = 1$ tangentially:

$$\lim_{t\to\infty} \arg(1 - F_t(z)) = \pm \frac{\arg \mu}{\alpha} = \pm \frac{\pi}{2}.$$

At the same time no trajectory can converge to $z = 1$ strongly tangentially since $\alpha \ne 1$.

3 Covering theorems (inner conjugation).

3.1 Backward flow invariant domains.

Let $f \in G[1]$ be the generator of a semigroup $S = \{F_t\}_{t \geq 0}$ having the Denjoy-Wolff point $\tau = 1$, i.e.,

$$f(1) \left(:= \angle \lim_{z \to 1} f(z) \right) = 0$$

and

$$f'(1) \left(= \angle \lim_{z \to 1} \frac{f(z)}{z-1} \right) \geq 0.$$

Definition 27 . *We say that a simply connected domain $\Omega \subseteq \Delta$ is a backward flow invariant domain (BFID) for S if S can be extended to a group on Ω.*

Example:

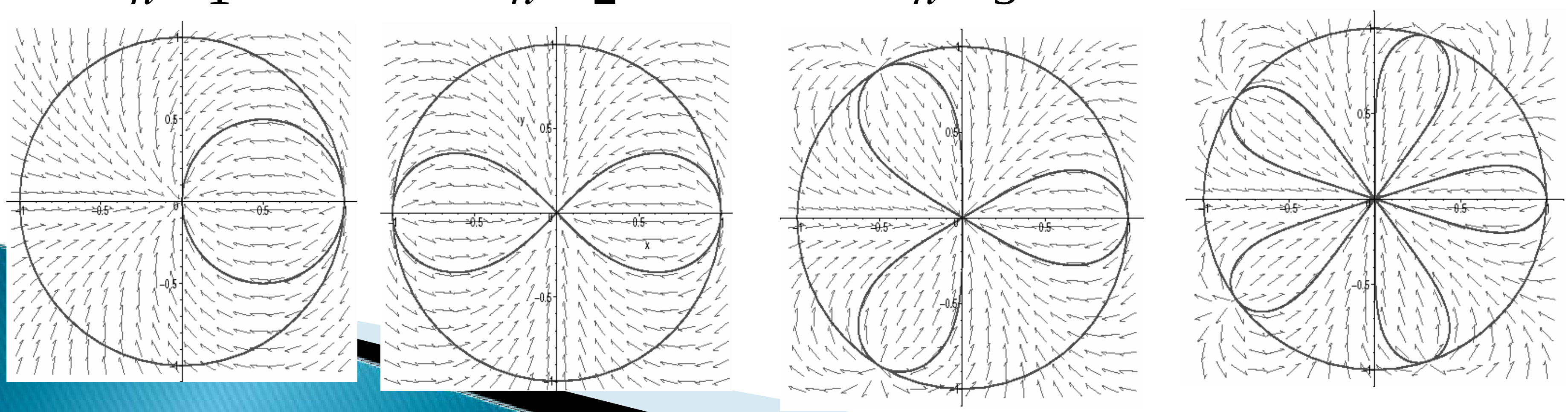
$$f(z) = z(1 - z^n) \qquad F_t(z) = \frac{ze^{-t}}{\sqrt[n]{1 - z^n + z^n e^{-nt}}}$$

$$\tau_0 = 0, \quad \tau_k = e^{i\frac{2\pi k}{n}}, \quad k = 1,2,\ldots,n$$ - the null points of f

$$f'(\tau_k) = -n \quad , \quad k = 1,2,\ldots,n$$

$$\varphi_k(z) = e^{i\frac{2\pi k}{n}} \sqrt[n]{\frac{1-z}{2}}$$

$n = 1$ $\qquad$ $n = 2$ $\qquad$ $n = 3$ $\qquad$ $n = 5$



Since Ω (if it exists) is simply connected one can find a Riemann conformal mapping $\varphi : \Delta \to \Delta$ such that $\varphi(\Delta) = \Omega$. It is clear that the family $\{G_t\}_{t\in\mathbb{R}}$ defined by

$$G_t(z) = \varphi^{-1}(F_t(\varphi(z))), \quad t \in \mathbb{R}, \tag{17}$$

forms a group of automorphisms of Δ.

Thus, the existence of BFID for S is equivalent to existence of an inner conjugation $\varphi : \Delta \to \Delta$,

$$F_t(\varphi(z)) = \varphi(G_t(z)), \quad t \geq 0 \tag{18}$$

where $\{G_t\}_{t\in\mathbb{R}}$ is a group on Δ.

Also it is clear that the group $\{G_t\}$ is not elliptic.

Thus the group $\{G_t\}_{t\in\mathbb{R}}$ is of either parabolic or hyperbolic type.

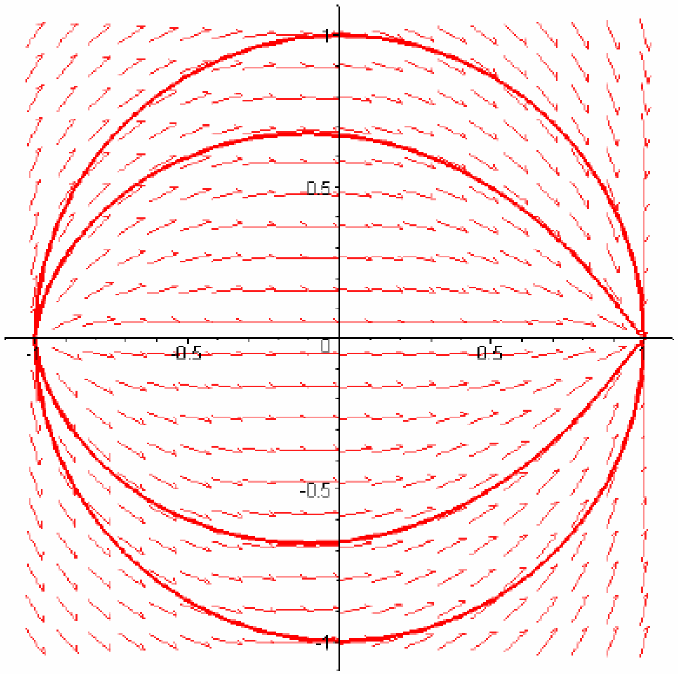
Definition 28 . *We say that* Ω *is of* p*-type if* $\{G_t\}_{t\in\mathbb{R}}$ *is parabolic and is of* h*-type if this group is hyperbolic.*

Remark 29 . *We will see below that if* Ω *is of* p*-type, then* $S = \{F_t\}_{t\geq 0}$ *also must be of parabolic type, i.e.,* $f'(1) = 0$. *At the same time if* Ω *is of* h*-type the semigroup* S *might be also of hyperbolic as well as parabolic type.*

Example 30 *Consider the holomorphic function f_1 defined by*

$$f_1(z) = -(1-z)^2 \cdot \frac{2+\sqrt{\frac{1+z}{1-z}}}{1+\sqrt{\frac{1+z}{1-z}}} \cdot \frac{1+z}{1-z}.$$

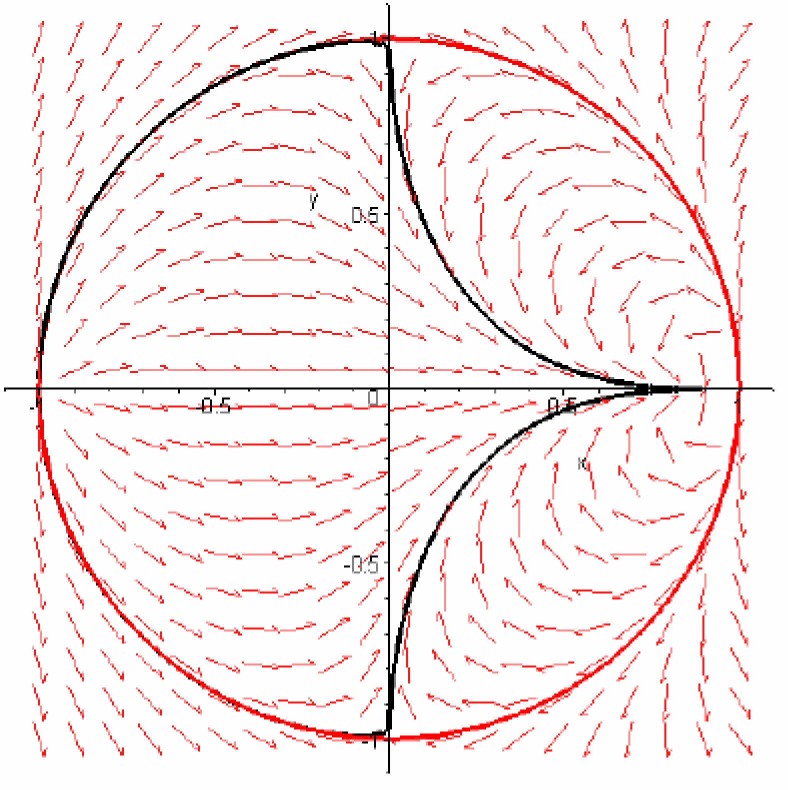
$$\varphi(z) = \frac{(w-1)^2 - 1}{(w-1)^2 + 1}, \quad \textit{where} \quad w = \sqrt{1+\sqrt{\frac{1+z}{1-z}}}.$$



Example 31 *Consider now another semigroup generator:*

$$f_2(z) = -(1-z)^2 \frac{1-z^2}{1+z^2}.$$

Since $f_2'(1) = 0$*, we conclude that the semigroup* $S_2 = \{F_t\}_{t \geq 0}$ *generated by* f_2 *is of parabolic type.*



Theorem 32 . *Let $f \in G[1]$ and let $S = \{F_t\}_{t \geq 0}$ be the semigroup generated by f. The following are equivalent:*

(i) there is a point $z \in \Delta$ such that $F_t(z) \in \Delta$ for all $t \in \mathbb{R}$ and the point $\zeta = \lim_{t \to -\infty} F_t(z)$ is a boundary regular null point of f, i.e., $f'(\zeta) = \angle \lim_{z \to \zeta} f(z)$ is finite;

(ii) the image $h(\Delta)$, where $h \in \Sigma[1]$ is defined by

$$h(z) = -\int_0^z \frac{dz}{f(z)}, \qquad (20)$$

contains a horizontal strip (or a horizontal half-plane);

(iii) there is a simply connected open subset $\Omega \subset \Delta$ such that $F_t|_\Omega$ is a group of automorphisms of Ω ;

(iv) there are real numbers a *and* b *such that the differential equation*

$$\left[a(z^2-1)+ib(1-z)^2\right]\varphi'(z)=f(\varphi(z)) \tag{21}$$

has a nonconstant solution $\varphi \in Hol(\Delta)$.

Moreover, in this case φ *is univalent and* $\Omega=\varphi(\Delta)$, *is a BFID for* S.

3.2 Classes $\Sigma^\alpha(G^\alpha)$ and corners of opening.

Now we will concenrate on geometric properties of a backward flow invariant domain Ω of p-type. In particular we show that under some smothness conditions Ω has a corner of opening (at the point $z = 1$) if and only if $f \in G^\alpha[1]$ (respectively, $h \in \Sigma^\alpha[1]$) for some $\alpha \in (1, 2]$.

We have proved that if Ω is of p-type, then for all $w \in \Omega$

$$\lim_{t \to -\infty} F_t(w) = 1. \tag{22}$$

As we have already mentioned, it follows by Theorem 2.4 in M. D. Contreras and S. Díaz-Madrigal, Analytic flows on the unit disk: angular derivatives and boundary fixed points, that if for at least one $w \in \Delta$ the trajectory $\{F_t(w)\}_{t \in \mathbb{R}}$ is well defined and condition (22) holds, then there is a horizontal half-plane contained in $h(\Delta)$. Thus condition (22) is necessary and sufficient condition for existence of BFID Ω of p-type.

Proposition 34 . *Let* $f \in G^{\alpha}[1]$ *with*

$$\lim_{z \to 1} \frac{f(z)}{(1-z)^{1+\alpha}} = a \neq 0, \infty$$

and let $S = \{F_t\}_{t \geq 0}$ *be the semigroup generated by* f. *Assume that for a point* $z \in \Delta$ *the trajectory* $\{F_t(z)\}_{t \geq 0}$ *can be extended to the whole real axis and*

$$\lim_{t \to -\infty} (1 - F_t(z)) = 1.$$

Then $\alpha \in [1, 2]$ *and*

(i) $|\arg(-a)| = \frac{\pi}{2}(2 - \alpha)$;

(ii) the trajectory $\{F_t(z)\}_{t<0}$ *converges tangentially to the point* $z = 1$ *as* t *goes to* $-\infty$. *Moreover, if* $\arg(-a) \neq 0$, *then*

$$\lim_{t \to -\infty} \arg(1 - F_t(z)) = (sign \arg(-a)) \frac{\pi}{2};$$

(iii) there is a BFID $\Omega \subset \Delta$ *of* p*-type which has a corner of opening* $\pi\gamma$ *at the point* $z = 1$ *with* $\gamma = \dfrac{1}{\alpha}$.

Corollary 35 . *If conditions of proposition 34 hold, then*

$$\lim_{t\to\infty} \arg(1 - F_t(z)) = -sign\,\arg(-a) \cdot \frac{\pi(2-\alpha)}{2\alpha}.$$

Thus, the trajectory $\{F_t(z)\}_{t\geq 0}$ *is tangential to the unit circle at the point* $z = 1$ *if and only if* $\alpha = 1$.

This trajectory is tangential to the real axis at the point $z = 1$ *if and only if* $\alpha = 2$.

Corollary 36 . *Let* $h \in \Sigma^{\alpha}[1]$ *with* $\mu = \lim(1-z)^{1+\alpha}h'(z)$. *The following assertions are equivalent:*

(i) there is $z \in \Delta$ *such that the point* $h(z) + t \in h(\Delta)$ *for all* $t \in \mathbb{R}$ *and*

$$\lim_{t \to -\infty} h^{-1}(h(z) + t) = 1;$$

(ii) $h(\Delta) \supset k(\Delta)$ *(a half-plane), where*

$$k(z) = \frac{ibz}{1-z} + c,$$

for some $b, c \in \mathbb{R}$.

Moreover, in this case:

(a) $\alpha \in [1,2]$*;*

(b) if $\alpha \in [1,2)$*, then there is no horizontal half-plane* $H \subset h(\Delta)$ *such that* $H \cap k(\Delta) = \emptyset$ *and* $b \cdot \arg \mu > 0$.

In a sense a converse assertion is also holds.

Theorem 37 . *Let* $h \in \Sigma[1]$ *and let* $F_t(z) = h^{-1}(h(z) + t)$ *for some* $z \in \Delta$ *can be extended to the whole real axis and*

$$\lim_{t \to -\infty} F_t(z) = 1.$$

In other words, there is a nonempty BFID Ω *of* p*-type. Suppose that* $\partial\Omega$ *has a Dini-smooth corner of opening* $\pi\gamma$ *at the point* $z = 1$*. The following assertions hold:*

(i) $h \in \Sigma_A^\alpha[1]$, *with* $\alpha = \dfrac{1}{\gamma}$ *and* $\mu := \angle\lim_{z\to 1}(1-z)^{1+\alpha}h'(z) \neq 0, \infty$;

(ii) $\frac{1}{2} \leq \gamma \leq 1$, *hence the corner of openning cannot be less than* $\dfrac{\pi}{2}$ *and* $|\arg\mu| \leq \dfrac{\pi(2\gamma-1)}{2\gamma}$;

(iii) the angular limit of the inverse Visser-Ostrovskii quotient

$$\angle\lim_{z\to 1}\frac{h(z)}{(z-1)h'(z)}$$

exists and equal γ.

THANKS FOR YOUR ATTENTION