The Global Geometry of SLE

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What is SLE?

F. & Kalkkinen (2003)

The correlator $\langle\mathfrak{O}\rangle_{\gamma[0,t]}$ can be recognised as a section of a certain bundle $\mathfrak{L}_h$ over the moduli space of Riemann surfaces.[6] In this context it is clear that $\mathcal{P}_X(\gamma)$ is consequently a holonomy, or a Wilson line, of this section when parallel transported from the fibre at X to the fibre over $X \setminus \gamma$ with respect to the connection $\mathrm{d} + T$.

Recall that if the correlator $\langle\mathfrak{O}\rangle$ satisfies $\hat{H}\langle\mathfrak{O}\rangle = 0$, the operator creates a state in the Verma module $V_{2,1}$. This module is closed under Virasoro action, which in turn is generated by the stress-energy tensor T. Since the Loewner process involves only insertions of the stress-energy tensor in the correlator, the final correlator $\langle\mathfrak{O}\rangle_{\gamma[0,t]}$ has to be that of an operator belonging to the same Verma module and satisfying the same differential equation. This is true irrespective of the moduli of the Riemann surface, and provides indeed an independent analytic characterisation of the correlators $\langle\mathfrak{O}\rangle_{\gamma[0,t]}$ as those sections of $\mathfrak{L}_h$ that are annihilated by $\hat{H}$.

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The CFT analysis leads us to consider sections of the line bundle $\mathfrak{L}_h$, which is a twisted version of the standard determinant bundle defined on the moduli space $\mathcal{M}_{g,1}$. By using the above defined projection π, we can construct the pull-back bundle $\pi^*\mathfrak{L}_h$ on $\widehat{\mathcal{M}}_{g,1}$. This bundle carries now a transitive Virasoro action, and can be equipped with a flat connection $\nabla = \mathrm{d} + L_{-1}$. In this way the Der $\mathcal{K}$ action is lifted to a Virasoro action in the quantum theory. The generator of the Loewner process

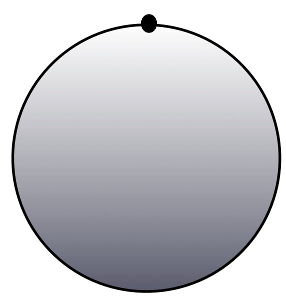
$$\hat{H} := \frac{\kappa}{2} L_{-1}^2 - 2L_{-2} \tag{71}$$

should therefore be seen naturally as a map

$$\hat{H} : \Gamma(\pi^*\mathfrak{L}_h) \longrightarrow \Gamma(\pi^*\mathfrak{L}_{h+2}) \ . \tag{72}$$

The relevant example for „standard“ SLE

- Let X be the Riemann sphere $\hat{\mathbb{C}} = \mathbb{P}^1(\mathbb{C})$ with $x \in X$, a marked point. Let $\mathcal{O}_x$ be the completion of the local ring at x, (the stalk of the structure sheaf at x).
- $\mathbb{D}_x := \operatorname{Spec} \mathcal{O}_x$ non-canonically isomorphic to $\mathbb{D}$
- choose a formal co-ordinate t_x at x, i.e., a topological generator of the maximal ideal $\mathfrak{m}_x$ of $\mathcal{O}_x$.



Interesting point: infinity ∞
Generalisation: abstract "half-disc", i.e., disc with an involution.

The group Aut(O)

- $\mathcal{O}$: completed topological $\mathbb{C}$-algebra $\mathbb{C}[[z]]$, with resp. to the natural filtration (Krull topology)
- $\text{Aut}(\mathcal{O})$: the group of continuous automorphisms of $\mathcal{O}$.
- $\text{Aut}(\mathcal{O}) \simeq a_1 z + a_2 z^2 + \ldots$, with $a_1 \in \mathbb{C}^*$ (formal power series).

$$\begin{array}{cc} \text{Aut}_+(\mathcal{O}) & \text{Der}_+(\mathcal{O}) = z^2\mathbb{C}[[z]]\partial_z \\ \cap & \cap \\ \text{Aut}(\mathcal{O}) & \text{Der}_0(\mathcal{O}) = z\mathbb{C}[[z]]\partial_z \\ & \cap \\ & \text{Der}(\mathcal{O}) = \mathbb{C}[[z]]\partial_z \end{array}$$

Associated spaces: power series development at infinity:

$$\begin{aligned} \text{Aut}(\mathcal{O}_\infty) &:= \left\{ \, bz + b_0 + \frac{b_1}{z} + \cdots \; , \quad b \neq 0 \, \right\} . \\ \text{Aut}_+(\mathcal{O}_\infty) &:= \left\{ \, z + b_0 + \frac{b_1}{z} + \cdots \; , \, \right\} . \end{aligned}$$

Proposition 1 (TUYKN).

1. $\mathrm{Aut}(\mathcal{O}) = \mathbb{C}^* \ltimes \mathrm{Aut}_+(\mathcal{O})$, *semi-direct product of* $\mathbb{C}^*$ *and* $\mathrm{Aut}_+(\mathcal{O})$. $\mathrm{Aut}(\mathcal{O})$ *acts on itself by composition.*

2. $\mathrm{Aut}(\mathcal{O})_+$ *is a pro-algebraic group, i.e.* $\mathrm{Aut}(\mathcal{O})_+ = \varprojlim \mathrm{Aut}_+(\mathcal{O}/\mathfrak{m}^n)$

3. *The exponential map* $\exp : \mathrm{Der}_+(\mathcal{O}) \to \mathrm{Aut}_+(\mathcal{O})$, *is an isomorphism.*

The infinite Kähler manifold of univalent functions

Definition 1. $\mathcal{M} := \{ f \in \mathcal{O}(\mathbb{D}) \mid$ *conformal, with* $f(0) = 0$ *and* $f'(0) = 1 \}$, *the set of* **univalent functions**.

$\mathcal{M}$ is a subset of the semi-direct product $\mathbb{R}_+ \ltimes \mathrm{Aut}_+(\mathcal{O})$, where

$$\mathrm{Aut}_+(\mathcal{O}) := z\left(1 + \sum_{k=1}^{\infty} c_k z^k\right) ,$$

and it is enough to study the traces in $\mathrm{Aut}_+(\mathcal{O})$. Now, this space has a natural affine structure with co-ordinates $\{c_k\}$, and the identity map corresponding to the origin 0.

By the De Branges-Bieberbach theorem one has $|a_n| \leq n$ and therefore $\mathcal{M}$ can be identified with an open subset of

$$\mathcal{M} \subset \prod_{n \geq 1} \mathrm{Ball}_{\mathbb{C}}(0, n+1)$$

Stochastic Löwner Equation

Definition 1 (Stochastic Lœwner Equation). *For* $z \in \mathbb{H}$, $t \geq 0$ *define* $g_t(z)$ *by* $g_0(z) = z$ *and*

$$\frac{\partial g_t(z)}{\partial t} = \frac{2}{g_t(z) - W_t}. \tag{1}$$

The maps g_t are normalised such that $g_t(z) = z + o(1)$ when $z \to \infty$ and $W_t := \sqrt{\kappa}\, B_t$ where $B_t(\omega)$ is the standard one-dimensional Brownian motion, starting at 0 and with variance $\kappa > 0$.

Itô form: $f_t(z) := g_t(z) - W_t$,

$$df_t(z) = \frac{2}{f_t(z)} dt - dW_t \ .$$

For a non-singular boundary point $x \in \mathbb{R}$, the generator A for the Itô-diffusion $X_t := f_t(x)$ is

$$A = 2\frac{1}{x}\frac{d}{dx} - \frac{\kappa}{2}\frac{d^2}{dx^2} \ .$$

Witt algebra / Virasoro algebra

Define first order differential operators:

$$\ell_n := -x^{n+1}\frac{d}{dx} \qquad n \in \mathbb{Z} \ ,$$

yields

$$A = \frac{\kappa}{2}\ell_{-2}^2 - 2\ell_{-1} \ .$$

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{\tilde{c}}{12}(n^3 - n)\,\delta_{n+m,0} \ ,$$

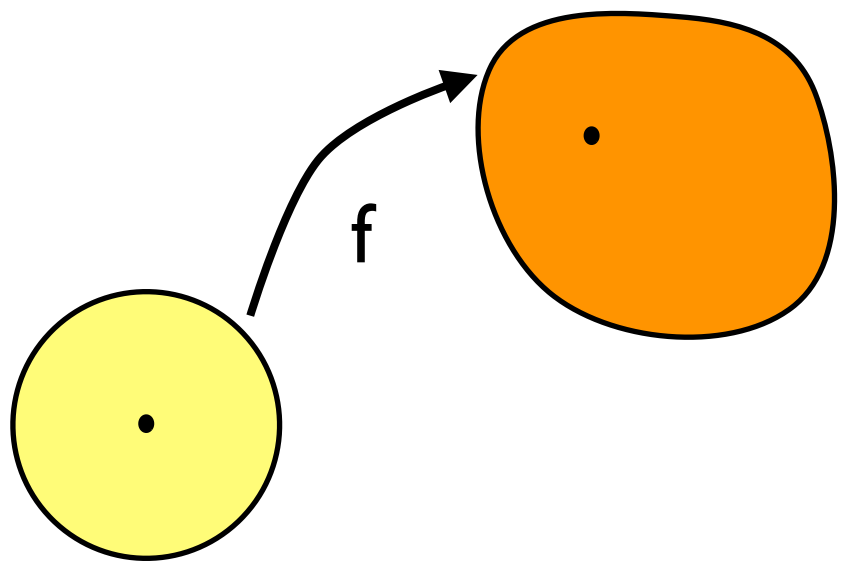
Determinant line bundles (regularised determinants)

To every Jordan domain one can associate the determinant of the Laplacian (with respect to the Euclidean metric and Dirichlet boundary conditions)

$$\det(\Delta_D) := \det(\Delta_{g_{\text{Eucl.}}})$$

trivial bundle over $\mathcal{M}$, where $f \in \mathcal{M}$ denotes the uniformising map from the unit disc $\mathbb{D}$ onto the domain D, containing the origin.

$$\begin{array}{c} \det(\Delta_{f(\mathbb{D})}) \\ \pi \downarrow \\ \mathcal{M} \end{array}$$



Polyakov`s conformal anomaly

Consider the space $\mathcal{F}$ of all flat metrics on $\mathbb{D}$ which are conformal to the Euclidean metric, obtained by pull-back.

For $f : \mathbb{D} \to D$ a conformal equivalence, define

$$\phi := \log |f'| \ .$$

which gives a correspondence of harmonic functions on $\mathbb{D}$ with with $\mathcal{F}$ via

$$ds = |f'||dz| = e^{\phi}|dz| \ .$$

To fix the SU$(1,1)$-freedom, we divide by it.

We work with the equivalence classes, so, e.g. 0 corresponds to the orbit of the Euclidean metric under SU$(1,1)$.

$$\det(\Delta_D) = e^{-\frac{1}{6\pi}\oint_{S^1}(\frac{1}{2}\phi * d\phi + \phi|dz|)} \cdot \det(\Delta_{\mathbb{D}})$$

Semi-group property

Consider the sequence of conformal maps between domains $\mathbb{D}, D, G$:

$$\mathbb{D} \xrightarrow{\ f\ } D \xrightarrow{\ g\ } G \ .$$

The relation of $\det(\Delta_G)$ and $\det(\Delta_{\mathbb{D}})$ is obtained via $\frac{d}{dz} g(f(z)) = g'(f(z)) \cdot f'(z)$, and

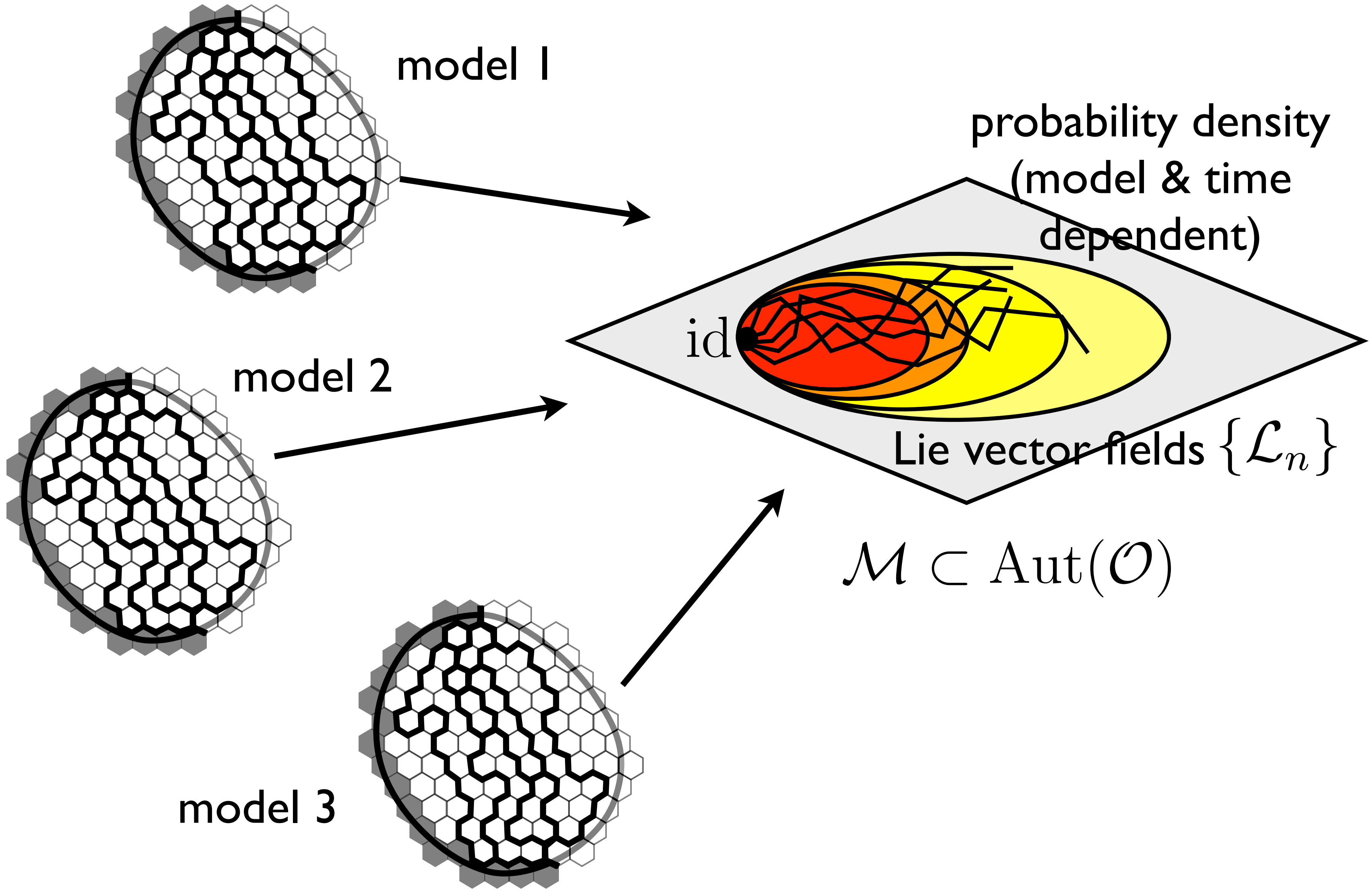
$$\log|g'(f(z)) \cdot f'(z)| = \underbrace{\log|g'(f(z))|}_{=:\psi(z)} + \underbrace{\log|f'(z)|}_{=:\phi(z)} \ .$$

The property of harmonic functions: $\oint_{S^1} \frac{1}{2}(\phi\partial_n\psi + \psi\partial_n\phi) = 0$, gives

$$\det(\Delta_G) = \underbrace{e^{-\frac{1}{6\pi}\oint_{S^1}(\frac{1}{2}\psi(f(z)){*}d\psi(f(z))+\psi(f(z))|dz|)}}_{I.} \cdot \underbrace{e^{\frac{1}{6\pi}\oint_{S^1}(\frac{1}{2}\phi{*}d\phi+\phi|dz|)} \cdot \det(\Delta_{\mathbb{D}})}_{II.}$$

where

$$\begin{aligned} I. &= \oint_{\partial D}(\frac{1}{2}\tilde{\psi}{*}d\tilde{\psi} + \tilde{\psi}|dw|) \qquad \text{with} \ \ \tilde{\psi}(w) := \log|g'(w)| \ , \\ II. &= \det(\Delta_D) \end{aligned}$$


model 1
model 2
model 3
probability density (model & time dependent)
id
Lie vector fields $\{\mathcal{L}_n\}$
$\mathcal{M} \subset \mathrm{Aut}(\mathcal{O})$

Virasoro algebra, Gelfand-Fuks and Weil-Petersson

The Virasoro algebra $\mathrm{Vir}_{\mathbb{C}}$ is spanned polynomial vector fields $e_n = -ie^{in\theta}\frac{d}{d\theta}$, $n \in \mathbb{Z}$, and $\mathfrak{c}$, with commutation relations $[\mathfrak{c}, e_n] = 0$ and

$$[e_m, e_n] = [e_m, e_n] + \omega_{c,h}(e_m, e_n) \cdot \mathfrak{c} \ ,$$

with the extended Gelfand-Fuks cocycle

$$\omega_{c,h}(v_1, v_2) := \frac{1}{2\pi} \int_0^{2\pi} \Big((2h - \frac{c}{12}) v_1'(\theta) - \frac{c}{12} v_1'''(\theta) \Big) \, v_2(\theta) \, d\theta \ ,$$

and v_1, v_2 being complex valued vector fields on S^1.

There exists a two-parameter family of Kähler metrics on this space, with the form at the origin

$$w_{c,h} := \sum_{k=1}^{\infty} \Big(2hk + \frac{c}{12}(k^3 - k) \Big) \, dc_k \wedge d\bar{c}_k \ ,$$

Analytic line bundles

- The Witt algebra has a representation in terms of the Lie fields $\mathcal{L}_{e_n}$ which act transitively on $\mathrm{Aut}_+(\mathcal{O})$.
- To have an action of $\mathrm{Vir}_{\mathbb{C}}$, one has to introduce a determinant line bundle.
- The line bundle $E_{c,h}$ is trivial, with total space $E_{c,h} = \mathrm{Aut}_+(\mathcal{O}) \times \mathbb{C}$. It is parametrised by pairs (f, λ), where f is a univalent function and $\lambda \in \mathbb{C}$.

It carries the following action

$$L_{v+\tau\mathfrak{c}}(f, \lambda) = (\mathcal{L}_v f, \lambda \cdot \Psi(f, v + \tau\mathfrak{c})) \ ,$$

where

$$\Psi_{c,h}(f, v + \tau\mathfrak{c}) := h \oint \left[\frac{wf'(w)}{f(w)}\right]^2 v(w)\frac{dw}{w} + \frac{c}{12} \oint w^2 S(f, w)\frac{dw}{w} + i\tau c \ ,$$

and where

$$S(f, w) := \{f; w\} := \frac{f'''(w)}{f'(w)} - \frac{3}{2}\left(\frac{f''(w)}{f'(w)}\right)^2 \ ,$$

The central element $\mathfrak{c}$ acts fibre-wise linearly by multiplication with ic.

Transitive action

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{C} & \longrightarrow & \mathrm{Vir}_{\mathbb{C}} & \longrightarrow & \mathrm{Witt} & \longrightarrow & 0 \\
\big\downarrow & & \big\downarrow & & \big\downarrow & & \big\downarrow & & \big\downarrow \\
0 & \longrightarrow & \mathbb{C} & \longrightarrow & \Theta_{E_{c,h}} & \longrightarrow & \Theta_{\mathcal{M}} & \longrightarrow & 0 \\
\big\downarrow & & \big\downarrow & & \big\downarrow & & \big\downarrow & & \big\downarrow \\
0 & \longrightarrow & \mathbb{C} & \longrightarrow & E_{c,h} & \longrightarrow & \mathcal{M} & \longrightarrow & 0
\end{array}$$

An **action** of a Lie algebra is a morphism from the Lie algebra $\mathfrak{g}$ to the tangent sheaf, and it is **transitive** if the map $\mathfrak{g} \otimes \mathcal{O}_X \to \Theta_X$ is point-wise surjective.

Algebraically, the situation corresponds to so-called **Harish-Chandra pairs** $(\mathfrak{g}, K)$.

Virasoro and Verma modules I

- The space of holomorphic sections $|\sigma\rangle \in \mathcal{O}(E_{c,h}) \equiv \Gamma(\mathrm{Aut}_+(\mathcal{O}), E_{c,h})$ of the line bundle $E_{c,h}$ carries a $\mathrm{Vir}_{\mathbb{C}}$-module structure.

- Let $\mathcal{P}$ be the set of (co-ordinate dependent) polynomials on $\mathcal{M}$, defined by
$$P(c_1, \ldots, c_N) : \mathrm{Aut}(\mathcal{O})/\mathfrak{m}^{N+1} \to \mathbb{C} \ ,$$
with $\mathfrak{m}$ the unique maximal ideal.

- $\mathcal{P}$ corresponds to the sections $\mathcal{O}(\mathcal{M})$ of the structure sheaf $\mathcal{O}_{\mathcal{M}}$ of $\mathcal{M}$ and it carries an action of the representation of the Witt algebra in terms of the Lie fields $\mathcal{L}_n \equiv \mathcal{L}_{e_n}$.

- In affine co-ordinates $\{c_n\}$, e.g.
$$\mathcal{L}_n = \frac{\partial}{\partial c_k} + \sum_{k=1}^{\infty}(k+1)c_k \frac{\partial}{\partial c_{n+k}} \qquad n \geq 1 \ .$$

Virasoro and Verma modules II

The action of $\mathrm{Vir}_{\mathbb{C}}$ on sections of $E_{c,h}$ can be written in co-ordinates:

$$
\begin{aligned}
L_n &= \partial_n + \sum_{k=1}^{\infty} c_k\, \partial_{k+n}, \quad n > 0 \\
L_0 &= h + \sum_{k=1}^{\infty} k\, c_k \partial_k \ , \\
L_{-1} &= \sum_{k=1}^{\infty} \left((k+2)c_{k+1} - 2c_1 c_k\right) \partial_k + 2hc_1 \ , \\
L_{-2} &= \sum_{k=1}^{\infty} \big((k+3)c_{k+2} - (4c_2 - c_1^2)c_k - a_k\big)\, \partial_k + h(4c_2 - c_1^2) + \frac{c}{2}(c_2 - c_1^2) \ ,
\end{aligned}
$$

where the a_k are the Laurent coefficients of $1/f$, and c the central charge.

A polynomial $P(c_1, \ldots, c_N) \in \mathcal{O}(E_{c,h})$ is a singular vector for $\{L_n\}$, $n \geq 1$, if

$$
\left(\partial_k + \sum_{k \geq 1} (k+1) c_k \partial_{k+n}\right) P(c_1, \ldots, c_N) = 0 \ .
$$

The highest-weight vector is the constant polynomial 1, and satisfies $L_0.1 = h \cdot 1$, $Z.1 = c \cdot 1$.

Hypo-ellipticity and sub-Riemannian geometry

By taking the projective limit we obtain the generator $\hat{A}_\infty$ of the flow on $\mathrm{Aut}_+(\mathcal{O}_\infty)$, corresponding to the Löwner equation for some fixed κ:

$$\hat{A}_\infty = \varprojlim \left(\frac{\kappa}{2} \frac{\partial^2}{\partial b_1^2} + 2 \sum_{k=2}^{N} P_k(b_1, \ldots, b_N) \frac{\partial}{\partial b_k} \right) ,$$

which is driven by one-dimensional standard Brownian motion, and where the polynomials P_k in the drift vector are defined on the coefficient body, with the $N \times N$ diffusion matrix

$$\begin{pmatrix} 1 & 0 & \ldots & 0 \\ 0 & 0 & & \\ \vdots & & \ddots & \vdots \\ 0 & & \ldots & 0 \end{pmatrix}$$

The generator of the diffusion process can be written in **Hörmander form** in terms of the tangent vector fields in the affine co-ordinates $\{b_k\}$,

$$L_1^\infty := \frac{\partial}{\partial b_1}, \qquad \text{and} \qquad L_2^\infty := -\sum_{k=2}^{\infty} P_k(\underline{b}) \frac{\partial}{\partial b_k} \ .$$

- Lift the process $f_\infty(t)$ on $\mathcal{M}$ induced by SLE, to the complex manifold $E_{c,h}$ by using sections σ. The sections have to be (projectively) flat with respect to the Hermitian connection $\nabla_{c,h}$ which determines c and h, for a given model.

- The random trajectories $\sigma_t := \sigma(f_\infty(t))$ should be (local) martingales. "Ensemble averages should be equal to time averages".

- We have to couple the parameters κ, c, h, which we shall obtain from the Doob-Getoor h-transform. Find harmonic sections, $\sigma_{\text{hr.}}$, then $\sigma(f_\infty(t))$ is a local martingale, for the lifted, now Virasoro generators $\hat{L}_n$.

- For polynomial sections, generated by the L_n, $n < 1$, by acting on the constant polynomial 1, the module contains a null vector, exactly if

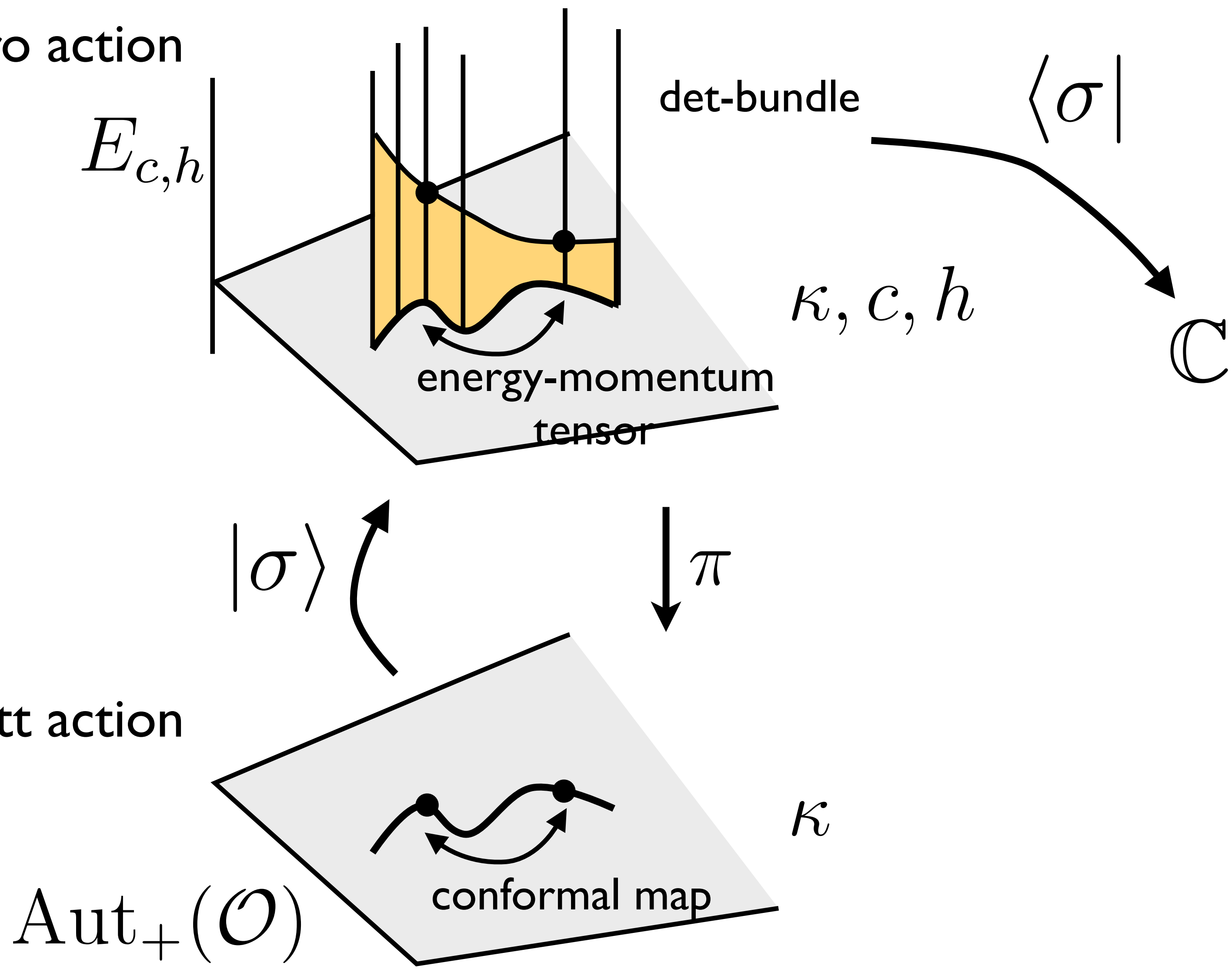
Theorem 1 (R. Fr.-W. Werner, M. Bauer and D. Bernard, 2002)

$$c_\kappa = \frac{(6-\kappa)(3\kappa-8)}{2\kappa} \qquad \textit{and} \qquad h_\kappa = \frac{6-\kappa}{2\kappa} \ .$$

- Therefore, all this polynomials are in the kernel of the lifted generator,

$$\frac{\kappa}{2}\hat{L}_1^2 - 2\hat{L}_2 \ ,$$

which acts as a differential operator.



The representation theoretic notion of “being degenerate at level two”, translates in probabilistic language into a generalised Doob h-transform.

References:

- R. Friedrich: A Renormalisation Group approach to Stochastic Lœwner Evolutions and the Doob h-transform (arXiv 2008)
- R. Friedrich: On Connections of Conformal Field Theory and Stochastic Lœwner Evolution (arXiv 2004)
- R. Friedrich: to come 2008 (I will put it onto the arXiv)
- Original sources: A. Kirillov, D. Yur'ev and Y. Neretin