Some semigroups of analytic functions arising from noncommutative probability

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The concept of independence plays a central role in probability theory. This concept distinguishes probability theory from the general measure theory. One of the remarkable features of quantum probability is the existence of several different notions of independence.

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In quantum probability the commutative algebra of random variables is replaced by a non-commutative algebra $\mathcal{A}$ of operators on a Hilbert space $\mathcal{H}$. The role of the probability measure is taken a unit vector $\xi \in \mathcal{H}$, which also is called the state. The probability distribution μ of quantum random variable $X \in \mathcal{A}$ is determined via the functional calculus of spectral theory by

$$\langle \xi, f(X)\xi \rangle \ = \int_{\sigma(X)} f(z) d\mu(z),$$

where $\sigma(X)$ is the spectrum of X.The measure μ is called also the spectral measure associated with the vector ξ and we can think of $\langle \xi, f(X)\xi \rangle$ as the expectation value of $f(X)$.

For a self-adjoint operator X the probability distribution μ is supported in $\mathbb{R}$. Non-compactly supported probability measures relate to unbounded self-adjoint operators. If X is positive operator, then $\sigma(X) \subset \mathbb{R}^{+} = \{x \in \mathbb{R} : x \geq 0\}$, and in the case of unitary operator X, we have $\sigma(X) \subset \mathbb{T} = \{\varkappa \in \mathbb{C} : |\varkappa| = 1\}$.

Denote by $\mathcal{M}(\mathbb{R})$ (similarly, $\mathcal{M}(\mathbb{R}^{+}), \mathcal{M}(\mathbb{T})$) the class of all probability measures on $\mathbb{R}$ (similarly, on $\mathbb{R}^{+}, \mathbb{T}$).

By $\mathcal{M}_0^2(\mathbb{R})$ we shall denote the class of all probability measures on $\mathbb{R}$ with finite variance and zero mean.

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In classical probability theory the distribution of the sum of two independent random variables is given by the convolution of the distributions of the two summands.

Many results of classical probability theory related to this operations. In this connection, the infinitely divisible probability distributions and one-parameter convolution semigroups occupy a central place in the theory.

Note, that infinitely divisible probability distributions are the natural object in the study of the limits of sums of independent random variables. One-parameter convolution semigroups are related to stationary processes with independent increments. The Lévy - Hinčin formula gives a description of the infinitely divisible distributions, and it shows that such distribution can be imbedding into a one-parameter convolution semigroup.

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In quantum probability there exist other notations of independence and convolution.
The first example of this kind is the concept of "freeness" which was introduced and investigated by Voiculescu.

In particular, he defined the free convolution for compactly supported measures on $\mathbb{R}$

*D. Voiculescu. Addition of certain non-commuting random variables. J. Funct. Anal.***66(3)***: 323–346, 1986.*

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This notion was extended by Maassen to measures with finite variance

H. Maassen. Addition of freely independent random variables.J. Funct. Anal. **106(2)***: 409–438, 1992,*

and to the whole class $\mathcal{M}(\mathbb{R})$ by Bercovici and Voiculescu

H. Bercovici and D. Voiculescu. Free convolution of measures with unbounded support. Indiana Univ. Math. J. **42(3)***: 733–773, 1993.*

We study infinite divisibility and embeddability property for monotone convolution, which was introduced by Muraki as additive monotone convolution for probability measures on $\mathbb{R}$ (the operation corresponds to the addition of monotone independent random variables)

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In the classical case analytic theory of convolution is based on the Fourier transform theory. Roughly speaking, using the Fourier transform one can converts convolution of probability measures into multiplication of characteristic functions.

In the non-commutative probability theory there are some kinds of transforms that convert additive and multiplicative monotone convolution of probability measures into composition of corresponding analytic functions.

In this connection it is natural to single out some semigroups of holomorphic mappings with respect to the operation of composition. The semigroups consist of analytic self-maps of the following domains

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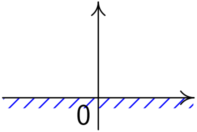
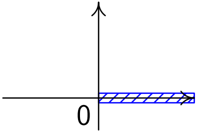
Basic domains

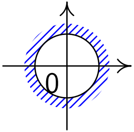
- Upper half-plane $\mathbb{U}$:



- Slit plane $\Delta = \mathbb{C} \backslash \mathbb{R}^+$:



- Unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$:



In an explicit calculation procedure for the free convolution as well as for the monotone convolution a central role is played by the Cauchy transform and reciprocal Cauchy transform.

The Cauchy transform of a probability measure μ on $\mathbb{R}$ is the analytic function G_μ on $\mathbb{U}$ defined by

$$G_\mu(z) = \int_{\mathbb{R}} \frac{d\mu(u)}{z-u}, \qquad z \in \mathbb{U}.$$

Note that μ is uniquely determined by G_μ and the inverse correspondence is given by the formula

$$\mu(B) = -\frac{1}{\pi} \lim_{\varepsilon \searrow 0} \int_B \mathrm{Im}\, G_\mu(x+i\varepsilon)dx,$$

valid for all Borel sets $B \subset \mathbb{R}$ for which $\mu(\partial B) = 0$.

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The reciprocal Cauchy transform F_μ is defined by

$$F_\mu(z) = 1/G_\mu(z), \qquad z \in \mathbb{U}$$

It is evident that F_μ maps $\mathbb{U}$ into itself. Denote by $\mathfrak{F}$ the following class of holomorphic self-maps of $\mathbb{U}$

$$\mathfrak{F} = \{f = F_\mu : \ \mu \in \mathcal{M}(\mathbb{R})\}.$$

Maassen received an intrinsic description of the class $\mathfrak{F}$. A holomorphic self-map $f : \mathbb{U} \to \mathbb{U}$ belongs to $\mathfrak{F}$ if and only if the angular derivative of f at the point at infinity is equal one. Therefore, $\mathfrak{F}$ is closed with respect to operation of composition and is a semigroup.

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Definition (Muraki)

Let μ and ν be two probability measures on $\mathbb{R}$ with reciprocal Cauchy transforms F_μ and F_ν. Then the **additive monotone convolution** *$\lambda = \mu \triangleright \nu$ of μ and ν is defined as the unique probability measure on $\mathbb{R}$ with reciprocal Cauchy transform $F_\lambda = F_\mu \circ F_\nu$.*

Denote by $\mathfrak{F}_0^2$ the class of reciprocal Cauchy transforms of probability measures μ on $\mathbb{R}$ with finite variance and zero mean

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It was obtain by Maassen that a holomorphic function $f : \mathbb{U} \to \mathbb{U}$ belongs to $\mathfrak{F}_0^2$ if and only if there exists a finite positive measure ν on $\mathbb{R}$ such that

$$f(z) = z + \int_{\mathbb{R}} \frac{d\nu(u)}{u - z}, \qquad z \in \mathbb{U}.$$

Note that $\mathfrak{F}_0^2$ was introduced as a class holomorphic self-maps of $\mathbb{U}$ with hydrodynamic normalization condition in our paper

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If μ is a probability measure on $\mathbb{R}^+$ one can define

$$\psi_\mu(z) = \int_{\mathbb{R}^+} \frac{uz}{1-uz} d\mu(u), \qquad z \in \Delta.$$

The function ψ_μ is called the moment generating function of μ.

Note that ψ_μ is analytic in Δ and never takes the value -1. Therefore we can consider the function

$$\eta_\mu(z) = \frac{\psi_\mu(z)}{1+\psi_\mu(z)}, \qquad z \in \Delta,$$

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Denote by $\mathfrak{T}$ the following class of holomorphic self-maps of Δ

$$\mathfrak{T} = \left\{ f = \eta_\mu : \ \mu \in \mathcal{M}(\mathbb{R}^+) \right\}.$$

An intrinsic description of the class $\mathfrak{T}$ is obtained by Belinschi and Bercovici

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A holomorphic function $f : \Delta \to \Delta$ belongs to $\mathfrak{T}$ if and only if $(\mathrm{Im} z)(\mathrm{Im} f(z)) \geq 0$ for $z \in \Delta$ and $\lim_{x \nearrow 0} f(x) = 0$. Thus $\mathfrak{T}$ is a semigroup with respect to operation composition.

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Let μ and ν be two probability measures on $\mathbb{R}^+$ with transforms η_μ and η_ν. Then the **multiplicative monotone convolution** *of μ and ν is defined as the unique probability measure $\lambda = \mu \circlearrowright \nu$ on $\mathbb{R}^+$ with transform $\eta_\lambda = \eta_\mu \circ \eta_\nu$.*

For a probability measure μ on $\mathbb{T}$ one can define

$$Q_\mu(z) = \int_{\mathbb{T}} \frac{\varkappa z}{1-\varkappa z} d\mu(\varkappa), \quad H_\mu(z) = \frac{Q_\mu(z)}{1+Q_\mu(z)}, \quad z \in \mathbb{D}.$$

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An element μ of semigroup $\mathcal{M}(\mathbb{R})$ is said to be $\triangleright$-infinitely divisible if for every natural number n there exists a measure $\mu_n \in \mathcal{M}(\mathbb{R})$ such that

$$\mu = \underbrace{\mu_n \triangleright \mu_n \triangleright \cdots \triangleright \mu_n}_{n \text{ times}}.$$

The concept of $\circlearrowright$-infinitely divisibility in $\mathcal{M}(\mathbb{R}^+)$ and $\mathcal{M}(\mathbb{T})$ is similarly defined.

Monotone infinitely divisible distributions play an important role in the theory of quantum stochastic processes whose increments are independent in the sense of monotone independence.

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The properties of monotone infinitely divisible distributions were studied by Belinschi, Bercovici, Franz, Muraki.
It appears that the monotone infinitely divisibility condition is equivalent to the condition of embeddability into a one-parameter monotone convolution semigroup.

A family $(\mu_t)_{t\geq 0}$ of measures μ_t in $\mathcal{M}(\mathbb{R})$ $(\mathcal{M}(\mathbb{R}^+), \mathcal{M}(\mathbb{T})$, respectively) is said to be the one-parameter semigroup if it is weakly continuous, $\mu_0 = \delta_0$ (Dirac measure) and $\mu_{t+s} = \mu_t \triangleright \mu_s$ $(= \mu_t \circlearrowright \mu_s$, respectively) for $s, t \geq 0$.

As well as in the classical case, characterizations of monotone infinitely divisible measures is given in terms of the corresponding transformations. A description of one-parameter composition semigroups gives a monotone analogue of Lévy-Khintchine formula.

One-parameter monotone convolution semigroups are in one-to-one correspondence with one-parameter composition semigroups.

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As well as in the classical case, characterizations of monotone infinitely divisible measures is given in terms of the corresponding transformations. A description of one-parameter composition semigroups gives a monotone analogue of Lévy-Khintchine formula.

One-parameter monotone convolution semigroups are in one-to-one correspondence with one-parameter composition semigroups.

The properties of monotone infinitely divisible distributions were studied by Belinschi, Bercovici, Franz, Muraki.
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One-parameter monotone convolution semigroups are in one-to-one correspondence with one-parameter composition semigroups.

Let $\mathfrak{P}$ be a composition semigroup of holomorphic self-maps of a domain $D \subset \mathbb{C}$. A family $(f^t)_{t\geq 0} \subset \mathfrak{P}$ is said to be the one-parameter semigroup in $\mathfrak{P}$ if $f^0(z) \equiv z$, $f^{t+s} = f^t \circ f^s$ for $s, t \geq 0$ and $f^t(z) \to z$ locally uniformly with respect to $z \in D$ as $t \searrow 0$.

The derivative

$$\frac{\partial}{\partial t} f^t(z)\Big|_{t=0} = v(z)$$

is a holomorphic function in D and is called the *infinitesimal generator* of the one-parameter semigroup $(f^t)_{t\geq 0}$ or an *infinitesimal transform* of the semigroup $\mathfrak{P}$. The infinitesimal generator v fully determines the one-parameter semigroup via the differential equation

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Denote by $\mathcal{K}(\mathfrak{P})$ the set of all infinitesimal transforms of semigroup $\mathfrak{P}$. A description of $\mathcal{K}(\mathfrak{P})$ is related to the infinitesimal structure of semigroup $\mathfrak{P}$.

Let f be in $\mathfrak{P}$. We will say that f is embeddable if there exists a one-parameter semigroup $(f^t)_{t\geq 0}$ in $\mathfrak{P}$ such that $f^1 = f$.

The class of all embeddable elements of a semigroup $\mathfrak{P}$ denote by $\mathcal{E}(\mathfrak{P})$. In context of quantum probability the class $\mathcal{E}(\mathfrak{P})$ is related to monotone infinitely divisible distributions.

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The following two results contain the infinitesimal structure of semigroups of reciprocal Cauchy transforms of probability measures on $\mathbb{R}$.

Theorem

A function v holomorphic in $\mathbb{U}$ belongs to $\mathcal{K}(\mathfrak{F})$ if and only if it admits a representation in the form

$$v(z) = \alpha + \int_{\mathbb{R}} \frac{1 + uz}{u - z} d\nu(u),$$

where $\alpha \in \mathbb{R}$ and ν is a finite, nonnegative Borel measure on $\mathbb{R}$.

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Recall that the composition semigroup $\mathfrak{F}$ is related to the additive monotone convolution semigroup $\mathcal{M}(\mathbb{R})$. Multiplicative monotone convolution semigroups $\mathcal{M}(\mathbb{R}^+)$ and $\mathcal{M}(\mathbb{T})$ are related to composition semigroups $\mathfrak{T}$ and $\mathfrak{L}$, respectively.

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It follows from classical paper of Loewner

K. Löwner, Untersuchungen über schlichte konforme Abbildungen des Einheitskreises I. Math. Ann. **89***: 103–121, 1923*

that a holomorphic function v in $\mathbb{D}$ belongs to $\mathcal{K}(\mathfrak{L})$ if and only if $v(z) = -zp(z)$, where p is holomorphic in $\mathbb{D}$ with nonnegative real part.

Theorem

A function v holomorphic in Δ belongs to $\mathcal{K}(\mathfrak{T})$ if and only if it admits a representation in the form $v(z) = zh(z)$, where h is holomorphic in Δ, $(\mathrm{Im} z)(\mathrm{Im} h(z)) \geq 0$ for $z \in \Delta$ and the integral

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The following two theorems give a complete characterization of $\triangleright$-infinitely divisible distributions in terms of solutions of Abel's functional equation.

Theorem

Let $f(z) \not\equiv z$ be in $\mathfrak{F}$. Then f belongs to $\mathcal{E}(\mathfrak{F})$ if and only if there exists a solution F of the functional equation

$$F \circ f(z) \;=\; F(z) + 1,$$

which is a holomorphic function in $\mathbb{U}$ satisfying the following conditions: $\mathrm{Im} F'(z) \leq 0$ for $z \in \mathbb{U}$ and $zF'(z) \to \infty$ as $z \to \infty$ (nontangent).

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$$\sup\left\{ \frac{y}{|F'(iy)|} \;:\; y > 0 \right\} \;<\; \infty.$$

For a description of $\circlearrowright$-infinitely divisible distributions on $\mathbb{R}^+$ it is necessary to characterize the set $\mathcal{E}(\mathfrak{T})$.

Before to formulate the result, we will note some properties of functions of class $\mathfrak{T}$.

Let $f(z) \not\equiv z$ belongs to $\mathfrak{T}$. Then $x \to f(x)$ is an increasing convex function on $(-\infty, 0)$ and $f(0-) = 0$. Therefore we have the alternative: either the equation $f(x) = x$ has unique negative root, or $f(x) \neq x$ for $x \in (-\infty, 0)$.
If $f(a) = a$ for $a \in (-\infty, 0)$, then by Schwarz's lemma $f'(a) \in (0, 1)$.

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Theorem

Let f be in $\mathfrak{T}$ and $f(x) \neq x$ for $x \in (-\infty, 0)$. Then f belongs to $\mathcal{E}(\mathfrak{T})$ if and only if there exists a solution F of the functional equation

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which is a holomorphic function in Δ and

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satisfies the following condition $(\mathrm{Im} z)(\mathrm{Im} h(z)) \geq 0$ for $z \in \Delta$.

Theorem

Let f *be in* $\mathfrak{T}$ *and* $f(a) = a$, $f'(a) = \gamma$, *where* $a \in (-\infty, 0)$, $\gamma \in (0, 1)$. *Then* f *belongs to* $\mathcal{E}(\mathfrak{T})$ *if and only if there exists a solution* F *of the functional equation*

$$F \circ f(z) \;=\; \gamma F(z),$$

which is a holomorphic function in Δ *and*

$$h(z) \;=\; \frac{aF(z)}{zF'(z)}$$

satisfies the following conditions: $h(a) = 0$, $h'(a) = 1$ *and* $(\mathrm{Im} z)(\mathrm{Im} h(z)) \geq 0$ *for* $z \in \Delta$.

As mentioned above, the class of $\circlearrowright$-infinitely divisible distributions on $\mathbb{T}$ is related to $\mathcal{E}(\mathfrak{L})$. A complete characterization of $\mathcal{E}(\mathfrak{L})$ was obtained in context of fractional iteration in our paper

M. Elin, V. Goryainov, S. Reich and D. Shoikhet. Fractional iteration and functional equations for functions analytic in the unit disk. CMFT. **2(2)***: 353–366, 2002.*

Theorem

Let $f(z) \not\equiv z$ be in $\mathfrak{L}$ and let $f'(0) = \gamma \neq 0$. Then f belongs to $\mathcal{E}(\mathfrak{L})$ if and only if there exists a solution F of the functional equation

$$F \circ f(z) \;=\; \gamma F(z)$$

which is a holomorphic function in $\mathbb{D}$ satisfying the following condition:

$$\frac{zF'(z)}{F(z)} \;=\; \frac{p(0)}{p(z)},$$

where p is holomorphic in $\mathbb{D}$, $\mathrm{Re} p(z) > 0$ for $z \in \mathbb{D}$ and

$$e^{-p(0)} = \gamma.$$

THANKS